

1. (26 pts) Evaluate the integral.

(a)  $\int \frac{2x^2 - 3x + 10}{x^3 + 5x} dx$

(b)  $\int \frac{1}{(x^2 - 1)^{3/2}} dx$

**Solution:**

(a)

$$\int \frac{2x^2 - 3x + 10}{x^3 + 5x} dx = \int \left( \frac{A}{x} + \frac{Bx + C}{x^2 + 5} \right) dx$$

Solve  $A(x^2 + 5) + x(Bx + C) = 2x^2 - 3x + 10$  to find the values  $A = 2$ ,  $B = 0$ , and  $C = 3$ .

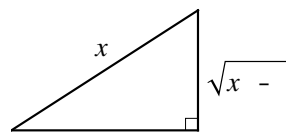
$$\int \frac{2x^2 - 3x + 10}{x^3 + 5x} dx = \int \left( \frac{2}{x} + \frac{3}{x^2 + 5} \right) dx$$

$$= 2 \ln|x| + \frac{3}{5} \arctan \frac{x}{5} + C_1$$

(b) Let  $x = \sec \theta$ ,  $dx = \sec \theta \tan \theta d\theta$ .

$$\begin{aligned} \int \frac{dx}{(x^2 - 1)^{3/2}} &= \int \frac{\sec \theta \tan \theta d\theta}{(\sec^2 \theta - 1)^{3/2}} = \int \frac{\sec \theta \tan \theta d\theta}{(\tan^2 \theta)^{3/2}} \\ &= \int \frac{\sec \theta \tan \theta d\theta}{\tan^3 \theta} = \int \frac{\sec \theta}{\tan^2 \theta} d\theta \\ &= \int \frac{\cos \theta}{\sin^2 \theta} d\theta = \int \csc \theta \cot \theta d\theta = \csc \theta + C \end{aligned}$$

$$= \frac{x}{\sqrt{x^2 - 1}} + C$$



2. (20 pts) This problem has three parts.

Let  $f(x) = 1 + \ln \frac{x}{x+1}$ . Consider the integral  $\int_1^4 f(x) dx$ .

- (a) Estimate the value of the integral using the trapezoidal approximation with 3 subintervals. Fully simplify your answer by combining logarithms.
- (b) Given that  $\frac{3}{4} f'''(x) < \frac{1}{50}$  for  $1 \leq x \leq 4$ , how large should  $n$  be to ensure that the approximation error for  $T_n$  is within  $10^{-4}$ ? Simplify your answer.
- (c) Is the approximation found in part (a) an underestimate or overestimate? Justify your answer (it is not necessary to find the exact value of the integral.)

**Solution:**

(a) Let  $\Delta x = \frac{b-a}{n} = \frac{3}{3} = 1$ . Then

$$\begin{aligned} T_3 &= \frac{1}{2} (\Delta x) (f(1) + 2f(2) + 2f(3) + f(4)) \\ &= \frac{1}{2} (1) \left( 1 + \ln \frac{1}{2} + 2 \left( 1 + \ln \frac{2}{3} \right) + 2 \left( 1 + \ln \frac{3}{4} \right) + 1 + \ln \frac{4}{5} \right) \\ &= \frac{1}{2} \left( 6 + \ln \frac{1}{2} \cdot \frac{2^2}{3^2} \cdot \frac{3^2}{4^2} \cdot \frac{4}{5} \right) = \frac{1}{2} \left( 6 + \ln \frac{1}{10} \right) \\ &= 3 - \frac{1}{2} \ln 10 \end{aligned}$$

(b) Let  $K = \frac{3}{4}$ , the maximum value of  $f'''(x)$ . Solve this inequality for  $n$ .

$$\begin{aligned} |E_{T_j}| &= \frac{K(b-a)^3}{12n^2} < 10^{-4} \\ \frac{(3-1)(3^3)}{12n^2} &< \frac{1}{10^4} \\ \frac{3^3}{4^2 n^2} &< \frac{1}{10^4} \\ n^2 &> \frac{3^3}{4^2} 10^4 \\ n &> \sqrt{\frac{3 \cdot 10^2}{4}} \\ n &> 75 \sqrt{3} \end{aligned}$$

(c) Because  $f'''(x) < 0$  on  $[1, 4]$ , the curve  $y = f(x)$  is concave down. The trapezoids all lie below the curve, so the approximation is an **underestimate**.

3. (30 pts) The following three problems are not related.

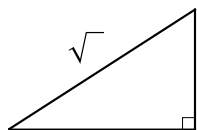
(a) Find the value of  $\int_1^{\sqrt{5}} \cot \cos^{-1} \frac{x}{\sqrt{5}} dx$ .

(b) Evaluate  $\int_0^1 6xe^{2x} dx$ . Justify any indeterminate limits.

(c) Does  $\int_1^{\infty} \frac{dx}{x(1+x^5)}$  converge or diverge? Justify your answer.

**Solution:**

(a) Let  $\theta = \cos^{-1} \frac{x}{\sqrt{5}}$ . Then  $\cos \theta = \frac{x}{\sqrt{5}}$ . A reference triangle shows that  $\sin \theta = \frac{2}{\sqrt{5}}$ , so  $\cot^{-1}(\frac{2}{\sqrt{5}}) = \sin^{-1}(\frac{2}{\sqrt{5}}) = \frac{\pi}{6}$ .



Note: Because  $\theta > 0$ , the angle is in the first quadrant.

(b) We will use integration by parts with  $u = x$  and  $dv = e^{2x} dx$ . Then  $du = dx$  and  $v = \frac{1}{2}e^{2x}$ .

$$\begin{aligned} \int_0^1 6xe^{2x} dx &= \lim_{t \rightarrow 1^-} \int_0^t 6xe^{2x} dx \\ &= \lim_{t \rightarrow 1^-} \left[ \frac{3xe^{2x}}{1} + \int_0^t \frac{3e^{2x}}{1} dx \right] \\ &= \lim_{t \rightarrow 1^-} \left[ 3xe^{2x} + \frac{3}{2}e^{2x} \right]_0^t \\ &= \lim_{t \rightarrow 1^-} \left( 3te^{2t} + \frac{3}{2}e^{2t} - \frac{3}{2} \right) \end{aligned}$$

4. (24 pts) Consider the region bounded by  $y = 4\sqrt{x}$ ,  $x = 0$ , and  $y = 1$ .

(a) Sketch and shade the region

(b) Set up but do not evaluate integrals to determine each of the following:

- I. The area of  $R$  using integration with respect to  $x$
- II. The area of  $R$  using integration with respect to  $y$
- III. The volume of the solid  $V$  is rotated about  $y = 1$  using the disk method.

**Solution:**

(a) Note that the curve  $y = 4\sqrt{x}$  intersects the line  $y = 1$  when  $4\sqrt{x} = 1 \Rightarrow x = \frac{1}{16}$ .

The curve can be represented as  $y = 4\sqrt{x}$ .

$$(b) \quad I. \quad A = \int_0^{\frac{1}{16}} (1 - 4\sqrt{x}) \, dx$$

$$II. \quad A = \int_0^1 \frac{y^2}{16} \, dy$$

$$III. \quad V = \int_a^b \pi r^2 \, dx = \int_0^{\frac{1}{16}} \pi (1 - 4\sqrt{x})^2 \, dx$$