1. [2360/072222 (30 pts)] Solve the initial value problem

- (j) **TRUE**. The change of variable y = 50 T yields $y^0 = 2y$.
- 3. (35 pts) The following parts are not related.
 - (a) (10 pts) Consider the initial value problem (IVP) ty^{\emptyset} 3(ln t)² $e^{-y} = 0$; $y(1) = \ln 8$.
 - i. (8 pts) Find the implicit solution to the IVP.
 - ii. (2 pts) Find the explicit solution to the IVP and state the interval over which the solution is valid.
 - (b) (25 pts) A particular solution to $L(\frac{t}{y}) = f$, where L is a linear operator, is $y_p = \cos t$. Suppose the characteristic equation for the associated homogeneous equation is $(r-2)(r^2-1) = 0$. Use Cramer's Rule to find the solution to the following initial value problem. No points for using other methods.

$$L({}^{\#}y) = f; \quad y(0) = 4; \ y^{0}(0) = 0; \ y^{0}(0) = 1$$

SOLUTION:

(a) i. The equation is separable.

Z
$$e^{y} dy = \frac{Z}{Z} 3 \frac{(\ln t)^{2}}{t} dt \quad (u = \ln t)$$

 $e^{y} = 3 \quad u^{2} du = (\ln t)^{3} + C$
 $e^{\ln 8} = (\ln 1)^{3} + C = C = 8$
 $e^{y} = (\ln t)^{3} + 8$

ii. The explicit solution is $y = \ln (\ln t)^3 + 8$. Clearly, t > 0 for input into the "inner" In t. For input into the "outer" natural logarithm function, we also need

$$(\ln t)^3 + 8 > 0 =) \ln t > \sqrt[9]{8} = 2 =) t > e^{-2}$$

The solution is valid on e^{-2} ; 7.

(b) Based on the characteristic equation, (r-2)(r+1)(r-1) = 0, the solution to the homogeneous equation is $y_h = c_1 e^{2t} + c_2 e^{-t} + c_3 e^t$ so the general solution to which we apply the initial conditions is $y = y_h + y_p$.

$$y(t) = c_1 e^{2t} + c_2 e^{-t} + c_3 e^t + \cos t$$

$$y^{0}(t) = 2c_1 e^{2t} - c_2 e^{-t} + c_3 e^t - \sin t$$

$$y^{0}(t) = 4c_1 e^{2t} + c_2 e^{-t} + c_3 e^t - \cos t$$

At t = 0 we have

Now use Cramer's Rule

The solution to the initial value problem is thus

$$y(t) = e^{2t} + e^{-t} + 3e^{t} + \cos t$$

- 4. [2360/072222 (29 pts)] The following parts are not related.
 - (a) (12 pts) Consider the function $f(t) = \begin{cases} t^2 & 0 & t < 0 \\ 0 & t < 2 \end{cases}$

i. (3 pts)

Thus

(b) i. Let $x_1(t)$; $x_2(t)$; $x_3(t)$ represent the mass (grams) of sugar and $V_1(t)$; $V_2(t)$; $V_3(t)$ the volume of solution (L) in Tank 1, 2, 3 at time t, respectively. Then with

$$\frac{dV}{dt}$$
 = flow rate in flow rate out

we have

For the amount of sugar in each tank, we will use

$$\frac{dx}{dt}$$
 = (flow rate in)(concentration in) (flow rate out)(concentration out)

$$\frac{dx_1}{dt} = (1)(2) + 4 \quad \frac{x_3}{100 + t} \quad 6 \quad \frac{x_1}{100 \quad t}$$

$$\frac{dx_2}{dt} = 6 \quad \frac{x_1}{100 \quad t} \quad 6 \quad \frac{x_2}{200}$$

$$\frac{dx_3}{dt} = 6 \quad \frac{x_2}{200} \quad 4 \quad \frac{x_3}{100 + t} \quad 1 \quad \frac{x_3}{100 + t}$$

$$\frac{2x_1^0 \cdot 3}{6x_2^0 \cdot 7} = \begin{cases} \frac{6}{6} & 0 & \frac{4}{100} & \frac{3}{100} & \frac{3}{2} &$$

(a)

$$\begin{pmatrix} 1 & 1 & 1 \\ 9 & 5 & 1 \end{pmatrix} = (1)(5) + 9 = \begin{pmatrix} 2 & 4 & +4 = (2)^2 = 0 \end{pmatrix} = 2$$
 with multiplicity 2

We need to find nontrivial solutions to $(\mathbf{A} \quad 2\mathbf{I})^{\mathbf{U}} = \mathbf{D}^{\mathbf{U}}$ giving

Since there is only one eigenvector, we need to find the generalized eigenvector by finding a nontrivial solution to $(\mathbf{A} \quad 2\mathbf{I})^{\frac{1}{\mathbf{U}}} = \mathbf{U}$.

The general solution is

$$(t) = c_1 e^{2t} \frac{1}{3} + c_2 e^{2t} t \frac{1}{3} + \frac{1}{4}$$

Applying the initial condition yields

$$c_1 \frac{1}{3} + c_2 \frac{1}{4} = \frac{2}{1}$$

with Cramer's Rule giving

and the final solution to the initial value problem as

$$\mathbf{x} = e^{2t} \quad \begin{array}{ccc} 2 & 7t \\ 1 & 21t \end{array}$$

- (b) We have $\text{Tr } \mathbf{A} = 2k; j\mathbf{A}j = k^2 + 2$ and $(\text{Tr } \mathbf{A})^2 4j\mathbf{A}j = 4k^2 4(k^2 + 2) = 8$
 - i. All real numbers Since $j\mathbf{A}j = k^2 + 2 \neq 0$ for all k, the system $\mathbf{A}^{\frac{1}{2}} = \mathbf{0}^{\frac{1}{2}}$ has only the trivial solution for all values of k. Thus, regardless of the value of k, the system will always have a unique equilibrium solution at (0,0).
 - ii. None $/A/ = k^2 + 2 > 0.4$