

Write your name below. This exam is worth 100 points. On each problem (except for problem 1),

2. (19 points) Consider the following matrix A

$$A = \begin{pmatrix} 2 & 0 & 3 & 0 & 2 \\ 6 & 1 & 2 & -3 & 0 \\ 0 & 2 & 5 & -4 & 3 \\ -3 & -4 & 7 & 0 & 5 \end{pmatrix}$$

- (a) (7 points) Find the permutation matrix P such that  $B := PA$  is symmetric. Show both P and B.
- (b) (12 points) Can B be factored as  $LDL^T$ ? If yes, find the factorization. If no, justify why it cannot be factored.

**Solutions:**

- (a) Note that the last three rows already look symmetric. Then we have

$$P = \begin{pmatrix} 2 & 0 & 3 & 0 & 2 \\ 6 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 5 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1 & 2 & -3 & 0 \\ 6 & 2 & 5 & -4 & 3 \\ 0 & -3 & -4 & 7 & 0 \\ 0 & 3 & 0 & 2 & 5 \end{pmatrix}$$

- (b) Note that since B is symmetric, it could be possible. According to the theorem, we need to know if B is regular. We attempt to perform LU factorization, first:

3. (20 points: 10 each)

The following two problems are unrelated.

(a) Determine if the following matrices are linearly independent

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}; \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix}; \begin{pmatrix} 1 & -5 \\ -4 & 0 \end{pmatrix} :$$

(b) Let  $V = \mathbb{R}^4$  and  $W \subseteq V$  be the space spanned by the vectors:

$\begin{pmatrix} 1 \\ -2 \\ 5 \\ -3 \end{pmatrix}$	;	$\begin{pmatrix} 1 \\ 3 \\ 1 \\ -4 \end{pmatrix}$	;	$\begin{pmatrix} 1 \\ 8 \\ -3 \\ -5 \end{pmatrix}$
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and the dimension of  $W$  is two.

(Solution 2) Another solution (albeit slightly harder) is to consider the matrix whose columns are the vectors and find it REF:

$$A = \begin{array}{cccc|c} \text{O} & 1 & 2 & 3 & 1 \\ \text{B} & -2 & 3 & 8 & \text{C} \\ \text{D} & 5 & 1 & -3 & \text{A} \\ \text{E} & -3 & -4 & -5 & \text{A} \end{array} \rightarrow \begin{array}{cccc|c} \text{O} & 1 & 2 & 3 & 1 \\ \text{B} & 0 & 7 & 14 & \text{C} \\ \text{D} & 0 & -9 & -18 & \text{A} \\ \text{E} & 0 & 2 & 4 & \text{A} \end{array} \rightarrow \begin{array}{cccc|c} \text{O} & 1 & 2 & 3 & 1 \\ \text{B} & 0 & 7 & 14 & \text{C} \\ \text{D} & 0 & 0 & 0 & \text{A} \\ \text{E} & 0 & 0 & 0 & \text{A} \end{array}$$

Here we are looking for the column space so we see that columns one and two are pivot columns and so select the pivot columns from the original matrix  $A$ :

$$\text{basis for } W = (1; -2; 5; -3)^T; (2; 3; 1; -4)^T$$

4. (19 points)

The following two questions are unrelated.

(a) (9 points) Let  $V = \mathbb{R}^3$  and  $W = \{(x; y; z)^T \in V : x^2 - 2xy + y^2 - z^2 = 0\}$ . Is  $W$  a vector subspace of  $V$ ? Prove or disprove.

(b) (10 points) Consider  $F(I)$ , the vector space of real valued functions on an interval  $I$ . Do the solutions to the differential equation

$$y'' + 5y' + 2y = 0$$

form a subspace of  $F(I)$ ? Prove that they do or show that they do not.

**Solution:**

(a)  $W$  is not a vector space as it is not closed under vector addition:

$$\text{Let } w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } w_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \text{ be members of } W, \text{ then}$$

$$w_1 + w_2 = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \text{ which is not a member of } W.$$

(b) This is a subspace. It is non-empty ( $y = 0$  is a solution) and closed under both scalar multiplication and vector addition:

For any  $c \in \mathbb{R}$  we have

$$(cy)'' + 5(cy)' + 2(cy) = c(y)'' + 5c(y)' + 2cy$$

$$= c(y'' + 5y' + 2y) = c(0) = 0.$$

For any 2 solutions to the differential equation we have

$$(y_1 + y_2)'' + 5(y_1 + y_2)' + 2(y_1 + y_2) = y_1'' + y_2'' + 5y_1' + 5y_2' + 2y_1 + 2y_2$$

$$= (y_1'' + 5y_1' + 2y_1) + (y_2'' + 5y_2' + 2y_2) = 0$$

5.