Write your name below. This exam is worth 100 points. On each problem (except for problem 1), you must show all your work to receive credit on that problem. You are NOT allowed to use your notes, book, calculator, or any other electronic devices.

Name:

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1. (21 points: 3 each) If the statement is *always true* mark "TRUE"; if it is possible for the statement to be false then mark "FALSE". No justification is necessary.

   (a) A 4×4 matrix with the only distinct eigenvalues being $-1, 0, 6$ must be diagonalizable.

   (b) Every non-trivial vector subspace has an orthonormal basis.

   (c) Let $\|\cdot\|$ be a norm coming from a real inner product $< \cdot, \cdot >$. If $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ then x and y must be orthogonal.

   (d) Let $< \cdot, \cdot >$ be an inner-product on a complex vectors space. Then $< w, v > = < v, w >$.

   (e) A matrix whose columns form an orthogonal basis of $\mathbb{R}^n$ is orthogonal.

   (f) Let $W \subseteq V$ where V is a vector space with $W = \text{span}\{v_1, v_2, v_3\}$ and inner product $< \cdot, \cdot >$. The orthogonal projection of a vector $v \in V$ is always given by

   $$Pv = < v, v_1 > \frac{v_1}{\|v_1\|^2} + < v, v_2 > \frac{v_2}{\|v_2\|^2} + < v, v_3 > \frac{v_3}{\|v_3\|^2}.$$

   (g) Let A be a square matrix with $\det(A) = 0$ then zero must be an eigenvalue of A.

2. (20 points) Let $C = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & -2 \\ -2 & -2 & 7 \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$.

   (a) (10 points) Find the associated quadratic form $q(x, y, z) = x^T C x$. Show that this quadratic form is positive definite.

   (b) (10 points) Find the Gram matrix K matrix corresponding to v_1, v_2, v_3 using the inner product $\langle x, y \rangle = x^T C y$. Does K have null directions? If yes, find them. If no, justify why. (For this part you may assume that $C > 0$).

3. (20 points)

Let $A = \begin{pmatrix} 0 & 2 & -2 \\ 1 & 1 & 0 \\ 4 & -4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 & -1 \\ 1 & 2 & -1 \\ -1 & 0 & 3 \end{pmatrix}$.

(a) (6 points) Find the eigenvalues of A and their algebraic multiplicities.

(b) (8 points) B has eigenvalue 2 with algebraic multiplicity of 2 and eigenvalue 4 with algebraic multiplicity of 1. Find all the eigenvectors of B and show that it is a complete matrix.

(c) (6 points) What are the matrices S and D such that $B = SDS^{-1}$? You do not need to calculate S^{-1}.

4. (20) points. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ and $K = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

   (a) (5 points) Find a basis for the image of A.

   (b) (9 points) Using the inner product defined by $\langle x, y \rangle = x^T K y$, find an orthogonal basis for the image of A. You may assume K is positive definite.

   (c) (5 points) Using the same inner product, find an orthonormal basis for the image of A.

5. (19 points) Find a symmetric orthogonal matrix P whose rst row is $\frac{1}{3}; \frac{2}{3}; \frac{2}{3}$: