APPM 3570 / STAT 3100
Fall 2021
Exam 2
November 10

This exam has two parts, and you may start on either as long as you follow the instructions for each.

Notes, your text and other books, cell phones, and other electronic devices are not permitted, except for calculators.

Calculators are permitted.

Write your name and sign and date your pledge to the CU Honor Code in the box below.

On my honor as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on either part of this work.

Name (Last, First): ______________________________

Signature: ______________________ Date: ____________

2. (4 points.) Suppose you play a game with a biased coin, which has a probability of observing heads equal to $2/3$. The probability of observing tails is $1/3$. You win \$10 if the coin lands on tails, and you lose \$6 if the coin lands on heads. What's the variance of the dollars won? ("Dollars won" can be negative.)

   Solution: $E[X^2] = \frac{172}{3}$ or 57.333; $E[X] = -\frac{2}{3}$, so $Var(X) = \frac{512}{9}$ or 56.889.

3. (4 points.) If $X \sim Binomial(n = 100, p = 1/5)$, which of the following statements is FALSE?

   Solution: $E(X) = np = 20$; $V(X) = np(1-p) = 16$; $(X-20)/4 = (X - E(X))/\sqrt{V(X)}$ has approximately a standard Normal distribution; $P(X = 50) = \binom{n}{50} p^{50}(1-p)^{n-50} = \binom{100}{50}\frac{4^{50}}{5^{100}}$; and $P(X = 0) = \binom{n}{0} p^{0}(1-p)^{n-0} = \left(\frac{4}{5}\right)^{100} \neq \frac{1}{5^{100}}$.

4. (4 points.) Suppose the number of hits a web site receives in any time interval is a Poisson random variable. If a particular site gets on average 5 hits per second, what's the probability it will get no hits in an interval of two seconds?

   Solution: $X \sim Poisson(10)$; $P(X = 0) = e^{-10}$.

5. (4 points.) The average number of acres burned by forest and range fires in a Colorado county is 700 acres per year, with a standard deviation of 360 acres. If the number of

Solution: For $y > 0$: $f_Y(y) = \int_{-\infty}^{\infty} f(x;y)\,dx = \int_{y/5}^{2y} 2e^{-x-y}\,dx = 2e^{-y}(-e^{-x})|_{x=y/5}^{x=2y} = 2(e^{-6y/5} - e^{-3y})$.

10. (4 points.) The joint p.m.f. of the random variables $X; Y; Z$ is:

$$P(X = x; Y = y; Z = z) = \frac{1}{4}; \text{ if } (x;y;z) = (1;2;3); (2;1;1); (2;2;1); \text{ or } (2;3;2):$$

What's the conditional probability $P(XYZ = 2 \mid Z = 1)$?

Solution: $P(XYZ = 2 \mid Z = 1) = \frac{P(XYZ=2;Z=1)}{P(Z=1)} = \frac{1/4}{1/2} = \frac{1}{2}$:

* Two More Questions Ahead! *

INSTRUCTIONS FOR PART II: Use the front _____ of your bluebook to answer to the following 2 questions. On the front cover of your bluebook, write (i) your name;

So $f(x,y) = 15xy^2 \neq 5x^4 \cdot \frac{15y^2(1-y^2)}{2} = f_X(x) \cdot f_Y(y)$, for $x,y \in (0,1)$, hence X and Y cannot be independent.

Solution II. In principle, X may take any value between 0 and 1. However, from the joint p.d.f. if $Y = y$ then $y < X$, i.e., knowledge of Y affects how X behaves, hence X and Y cannot be independent. (A similar argument follows from observing how X affects Y.)

Problem B. (30 points.) Let X and Y be discrete random variables with joint probability mass function (p.m.f.):

$$p(x,y) = \frac{1}{e \cdot (x+1)!}\left(1 - \frac{1}{x+1}\right)^{y-1}; \text{ for integers } x \geq 0 \text{ and } y \geq 1.$$

(a) Show that X has a Poisson distribution with parameter $\lambda = 1$.
Hint. $\sum_{k=0}^{\infty} \lambda^k = \frac{1}{1-\lambda}$

case if X and Y were independent.

Solution II.

$$P(X=1;Y=1) = p(1;1) = \frac{1}{2e};$$

$$P(X=1) = \frac{1}{e};$$

$$P(Y=1) = \sum_{x=0}^{\infty} p(x;1) = \frac{1}{e}\sum_{x=0}^{\infty}\frac{1}{(x+1)!} = \frac{1}{e}\left(\sum_{k=0}^{\infty}\frac{1}{k!} - \frac{1}{0!}\right) \quad e - 1 \dots$$