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A Multiresolution Strategy for Reduction of Elliptic PDEs and Eigenvalue Problems

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equations on a coarse scale that reproduces the solution on that scale. The difficulty, of course, is that such a solution is influenced by the finer scale behavior of the coefficients. Capturing the influence of fine scales (exactly or approximately) on the behavior of the solution on the coarse scale is the problem generally known as that of homogenization (although particular formulations may be rather different).

Typically such problems were addressed by using asymptotic methods or weak limits; see for example, [2, 7, 15, 22, 27, and references therein]. The basic limitation of these methods is that they require the fine scale behavior to be fairly well separated from the behavior on the coarser scales, so that small parameters may be found in the problem. Recently, a multiresolution strategy for homogenization has been proposed in [4]. Using the notion of multiresolution analysis (MRA), we consider the transition between two adjacent scales explicitly. Namely, one obtains an equation for the projection of the solution on the coarser scale. This procedure (the so-called reduction) may then be repeated over many scales and thus does not require the small parameter assumptions typical for asymptotic methods.

The basic step of the reduction involves computing a Schur complement. (The use of the Schur complement in multilevel methods is not new and plays a role in algebraic multigrid and domain decomposition methods (see, e.g., [8, 25]). Steinberg and McCoy in [26] use the Schur complement for multiresolution effective medium computations. Additionally, Knapek in [19] has used the Schur complement for a multigrid-based homogenization technique.) Two problems have to be addressed in order for the multiresolution strategy for homogenization to be a practical method. First, the transition between the two scales has to be computationally efficient. However, simply truncating the matrices as has been suggested in some of the references mentioned above is not satisfactory since there is no control of the quality of the approximation. Second, the form of equations has to be preserved so that one can use the reduction step in a recursive manner. By the ''form of the equations'' we understand either algebraic form or some alternative algebraic structure. The only requirement is that it may be used recursively. The meaning of this remark will become clear below.

In [4] the multiresolution strategy for reduction and homogenization has been

problems. First, the use of high-order wavelets permits us to develop efficient numerical methods for the reduction procedure. Second, we demonstrate that if the MRA is chosen correctly for a given problem, then the small eigenvalues of the reduced operators differ only slightly from those of the original operator. In particular, correctly here means that the basis must have a sufficient number of vanishing moments. As a result, we obtain a method for constructing a low-dimensional approximation to a multiscale elliptic problem such that this approximation accounts for both small eigenvalues and the corresponding eigenvectors. Computing these quantities is desirable in many applications, e.g., computational chemistry, although we do not address such problems here.

A model equation that we consider is of the form

$$0 , r(a(\mathbf{x}) ,) \quad f(\mathbf{x}) \tag{1.1}$$

with periodic boundary conditions. However, our method is applicable to other boundary conditions as well, in which case the wavelet basis has to satisfy the boundary conditions. For the eigenvalue problem we also assume that the ratio $\max a(\mathbf{x})/\min a(\mathbf{x})$ is moderate in size over the domain.

We are not familiar with prior numerical algorithms of the type presented in this paper. Although similar goals have been sought by multigrid methods (see [6, 19–21]), the approach and the results of this paper appear to be different. Papers on classical homogenization of elliptic eigenvalue problems (see, e.g., [17, 18]) also yield a different type of results.

We start in Section 2 by introducing notation and briefly reviewing related results on homogenization of elliptic equations. In Section 3 we address the problem of the multiresolution reduction for elliptic equations using high-order wavelets. We prove that the rate of the off-diagonal decay of the blocks of the reduced operator is preserved and is the same as that of the blocks of the non-standard form. Thus, the reduced operator is compressible in wavelet bases and, for a given accuracy, the sparsity is controlled by the number of vanishing moments of the wavelet basis. We demonstrate that the spectral bounds for the reduced operator on all scales are the same as those for the original operator. We obtain estimates for small eigenvalues which show that the reduced operator is better at preserving them than the projection of the original operator on the corresponding scale. We also introduce a modified reduction procedure to improve the accuracy of preservation of small eigenvalues. Finally, in Section 4 we present results of our numerical experiments.

2. PRELIMINARY CONSIDERATIONS

2.1. Notation

In this section we set our notation and give a brief description of the concept of multiresolution analysis and wavelets. For the details we refer to, e.g., [9]. As usual, we consider a chain of subspaces

$$r r r , \mathbf{V}$$

If

(in its general form). Consider a bounded linear operator $\mathbf{S}_j$: $\mathbf{V}_j$ r $\mathbf{V}_j$ together with an equation

$$\mathbf{S}_j x \quad f, \tag{2.3}$$

which we may write as

$$\begin{pmatrix} \mathbf{A}_{\mathbf{S}_j} & \mathbf{B}_{\mathbf{S}_j} \\ \mathbf{C}_{\mathbf{S}_j} & \mathbf{T}_{\mathbf{S}_j} \end{pmatrix} \begin{pmatrix} d_x \\ s_x \end{pmatrix} \quad \begin{pmatrix} d_f \\ s_f \end{pmatrix}. \tag{2.4}$$

Formally eliminating d_x from (2.4) by substituting d_x $\mathbf{A}_{\mathbf{S}_j}^{01}(d_f \ 0 \ \mathbf{B}_{\mathbf{S}_j} s_x)$ gives us

$(\mathbf{T}_{\mathbf{S}}$

original partial differential equation (2.7) has the same average or coarse-scale behavior as the solution of the partial differential equation with coefficients given by $a^h(\mathbf{x})$.

The approach of classical homogenization is to consider the family of equations

$$0 \, , \, \mathrm{r}(a(\mathbf{x}/\mathrm{e}) \, , \, u^{\mathrm{e}}(\mathbf{x})) \quad f(\mathbf{x}), \tag{2.8}$$

where the function $a(\mathbf{x})$ is periodic. Clearly, as e r 0, the coefficients $a(\mathbf{x})$ become more and more oscillatory. This implies that the coefficients change on a scale that is asymptotically fine relative to the fixed coarse scale of the solution. The problem is to find an equation of the form (2.7) which has the weak limit u^0 of u^{e} as its solution. The coefficients of this equation are taken to be the effective coefficients of the family of equations given by (2.7). Such formulations of homogenization problems are discussed in detail in, e.g., [2] and references therein.

Multiresolution homogenization as defined in [4] is, like classical homogenization, a limit process. It finds the effective coefficients of ODEs by (i) computing recurrence relations of the coefficients over one scale of reduction, (ii) finding the limit of the coefficients over infinitely many scales, and (iii) identifying an equation with smooth or constant coefficients such that reduction over infinitely many scales results in the same equation as the limit from (ii). This procedure does not assume asymptotic separation of fine and coarse scales.

In this paper we use the term homogenization to refer to a limit procedure. In classical homogenization, the fine scale is associated with a small parameter, and the limit is considered as this small parameter goes to zero. Multiresolution homogenization considers a limit over infinitely many scales. We use the term reduction to refer to an explicit transition between neighboring scales and in this paper study it over finitely many scales. We permit the coefficients to vary on intermediate scales.

The reduction procedure when applied to partial differential equations presents several interesting problems. First let us briefly describe some important points about the reduction procedure for ODEs. It is observed in [4] that, for systems of linear ordinary differential equations, using the Haar basis (and also multiwavelets with disjoint supports; see [1]) provides a technical advantage. Since the functions of the Haar basis on a fixed scale do not have overlapping supports, the recurrence relations for the coefficients and forcing terms in the equation may be written as local relations and solved explicitly. Thus for systems of ODEs, an explicit reduction *and* homogenization procedure is possible. Gilbert [11] has demonstrated the reduction and homogenization of [4] applied to the one-dimensional version of (2.7) and has established a connection to classical homogenization results (see, e.g., [2, 15]). Dorobantu [10] has also connected multiresolution homogenization with classical homogenization in the one-dimensional case.

The situation for partial differential equations is more complicated. Indeed, when the reduction procedure is applied to partial differential equations of the form, e.g., (2.7), the recurrence relations for the reduced operators do not appear to be locally solvable. Therefore, unlike the homogenization and reduction procedure outlined in [4] for ODEs, there does not seem to be an explicit local recurrence relation for the coefficients of the partial differential equation. Since such a recurrence does not appear feasible even with the Haar basis, one might as well consider the general scheme

outlined in [4], where high-order wavelets are used. We show in fact that there are compelling reasons for the use of such wavelets.

3.1. Preservation of Ellipticity

and

$$(\mathbf{R}_{\mathbf{S}_j}x,\ x) \quad \left\langle \mathbf{Z}^{*}\begin{pmatrix}\mathbf{A}_{\mathbf{S}_j} & \mathbf{B}_{\mathbf{S}_j}\\ \mathbf{B}^{*}_{\mathbf{S}_j} & \mathbf{T}_{\mathbf{S}_j}\end{pmatrix}\mathbf{Z}\begin{pmatrix}0\\ x\end{pmatrix},\begin{pmatrix}0\\ x\end{pmatrix}\right\rangle .$$

For the lower bound we obtain

$$(\mathbf{R}_{\mathbf{S}_j}x,\ x)$$

$$\left\langle \begin{pmatrix}\mathbf{A}_{\mathbf{S}_j} & \mathbf{B}_{\mathbf{S}_j}\\ \mathbf{B}^{*}_{\mathbf{S}_j} & \mathbf{T}_{\mathbf{S}_j}\end{pmatrix}\mathbf{Z}\begin{pmatrix}0\\ x\end{pmatrix},\ \mathbf{Z}\begin{pmatrix}0\\ x\end{pmatrix}\right\rangle \ \geq\ m(\|\mathbf{A}^{01}_{\mathbf{S}_j}\mathbf{B}_{\mathbf{S}_j}x\|^2\ /\ \|x\|^2)\ \geq\ m\|x\|^2. \quad (3.8)$$

To estimate the upper bound, we use $\mathbf{R}_{\mathbf{S}_j}\ /\ \mathbf{B}^{*}_{\mathbf{S}_j}\mathbf{A}^{01}_{\mathbf{S}_j}\mathbf{B}_{\mathbf{S}_j}\quad \mathbf{T}_{\mathbf{S}_j}$ and positive definiteness of $\mathbf{A}^{01}_{\mathbf{S}_j}$ to obtain

$$(\mathbf{R}_{\mathbf{S}_j}x,\ x)\ \leq\ (\mathbf{T}_{\mathbf{S}_j}x,\ x).$$

Since $\begin{pmatrix}\mathbf{A}_{\mathbf{S}_j} & \mathbf{B}_{\mathbf{S}_j}\\ \mathbf{B}^{*}_{\mathbf{S}_j} & \mathbf{T}_{\mathbf{S}_j}\end{pmatrix}$ satisfies the same spectral bounds as $\mathbf{S}_j$, we have

$$(\mathbf{T}_{\mathbf{S}_j}x,\ x)\ \leq\ M\|x\|^2.$$

This completes the proof. We note that since we have made no assumptions (other than orthogonality) about the multiresolution analysis, the properties (3.3) and (3.2) do not depend on dimension or the choice of wavelet basis.

The ellipticity estimate of (3.3) raises the important question of whether it is possible (and under which conditions) to have exactly or approximately the lower eigenvalues of $\mathbf{S}_j$ as eigenvalues of $\mathbf{R}_{\mathbf{S}_j}$

$$s^{j}_{k,k'} = \int\int K(x, y)\, \mathrm{f}_{j,k}(x)\, \mathrm{f}_{j,k'}(y)\, dxdy, \qquad (3.9d)$$

and $K(x, y)$ is the kernel of a Calderón–Zygmund or a pseudo-differential operator **T**. We assume that K satisfies the conditions

$$K(x, y) \quad ¡ \quad \frac{1}{\quad}$$

basic technique of the proof is the same as that of Tchamitchian in [28]. To indicate that the results of this paper are valid in higher dimensions, we state the two-dimensional version of Theorem 3.3 without proof:

THEOREM 3.4. *If a matrix* $\{m_{k,k',l,l'}\}_{k,k',l,l' \in \mathbf{Z}}$ *satisfies*

$$m_{k,k',l,l'} \quad C(1 + |k-k'| + |l-l'|)^{-(2+\alpha)} \tag{3.16}$$

(*where* $\alpha \in \mathbf{Z}$, $\alpha \geq 2$) *and if the matrix is invertible on* l^2, *then*

$$m^{-1}_{k,k',l,l'} \quad C\gamma(1 + |k-k'| + |l-l'|)^{-(2+\alpha)}. \tag{3.17}$$

Matrices which satisfy (3.16) also form an algebra under multiplication.

We use Theorems 3.3 and 3.4 to show that at all stages of the reduction procedure in both one and two dimensions the matrices representing the **A**, **B**, and **C** blocks of the reduced operators (2.6) satisfy the same off-diagonal decay estimate (3.13) as the blocks of the non-standard form in Theorem 3.2 and its two-dimensional analogue. In other words, the reduction procedure preserves sparsity for a wide class of elliptic operators. In this sense the form (or structure) is preserved under the reduction procedure, which allows us to apply it over a finite number of scales. The following theorem applies to the one-dimensional case, but analagous results for two dimensions can be proved using Theorem 3.4.

THEOREM 3.5 (Preservation of structure over finitely many scales). *Let us assume that the operator* **S** *and the wavelet basis satisfy conditions of Theorem* 3.2 *and, in addition,* **S** *is a self-adjoint, strictly elliptic operator. Let* $\mathbf{R}_j$ *be the reduced operator on some scale j, where reduction started at some scale n,* $n \geq j$, $n, j \in \mathbf{Z}$, *and let* $\mathbf{A}_{\mathbf{R}_j}$, $\mathbf{B}_{\mathbf{R}_j}$, *and* $\mathbf{C}_{\mathbf{R}_j}$ *be its blocks. Then the bi-infinite matrices* $\alpha^{r,j}$, $\beta^{r,j}$, *and* $\gamma^{r,j}$ *representing these blocks satisfy*

$$|\alpha^{r,j}_{k,l}| + |\beta^{r,j}_{k,l}| + |\gamma^{r,j}_{k,l}| \leq C^{n,j}_{M}(1 + |k-l|)^{-M-1}, \tag{3.18}$$

for all integers k, l.

Proof. Our starting point is the operator $\mathbf{S}_n$ and its blocks, $\mathbf{A}_{\mathbf{S}_n}$, $\mathbf{B}_{\mathbf{S}_n}$, $\mathbf{C}_{\mathbf{S}_n}$, and $\mathbf{T}_{\mathbf{S}_n} \quad \mathbf{S}_{n-1}$.

Matrices representing these blocks satisfy the estimate of Theorem 3.2. Since $\mathbf{S}_n$ is positive definite, so is $\mathbf{A}_{\mathbf{S}_n}$ (see Section 3.1), and thus $\mathbf{A}^{-1}_{\mathbf{S}_n}$ exists and, according to Theorem 3.3, satisfies the estimate in (3.13). Since $\mathbf{B}_{\mathbf{S}_n}$ and $\mathbf{C}_{\mathbf{S}_n}$ satisfy the same estimate (3.13), the product $\mathbf{C}_{\mathbf{S}_n}\mathbf{A}^{-1}_{\mathbf{S}_n}\mathbf{B}_{\mathbf{S}_n}$

The operator $\mathbf{S}_{n/1}$ is the projection on the scale $n / 1$ of the operator $\mathbf{T}$ and the operator $\mathbf{F}_{n/1}$ has fast decay and satisfies the estimate (3.13). The blocks $\mathbf{A}_{\mathbf{R}_{n/1}}$, $\mathbf{B}_{\mathbf{R}_{n/1}}$, $\mathbf{C}_{\mathbf{R}_{n/1}}$, and $\mathbf{T}_{\mathbf{R}_{n/1}}$ of the operator $\mathbf{R}_{n/1}$ may be written as a difference of the corresponding blocks of these two terms. Theorem 3.2 guarantees that the contribution from $\mathbf{S}_{n/1}$ has the proper decay. On the other hand, the contributions from $\mathbf{F}_{n/1}$ have at least the same rate of decay as $\mathbf{F}_{n/1}$ itself since the blocks are obtained by a wavelet transform.

We prove Theorem 3.5 by induction assuming that on some scale j we have

$$\mathbf{R}_j \quad \mathbf{S}_j \; 0 \; \mathbf{F}_j, \tag{3.22}$$

where

3.3. A Fast Method for Computing the Reduced Operator

In practical application of the reduction procedure (2.5) one of the critical issues is the cost of computing the reduced operator (2.6). The sparsity of the operators involved in the reduction is assured by Theorem 3.5. However, one still needs an algorithm for computing the reduced operator.

It turns out that a multiresolution LU decomposition algorithm may be used to obtain the reduced operator [12]. The multiresolution LU decomposition is performed with respect to the product of non-standard forms rather than the ordinary matrix product. It has complexity $O(N)$ for a fixed relative error ε and provides a direct solver for linear systems written using the non-standard form.

The algorithm in [12] provides an alternative to the computation of $\mathbf{A}_{s_j}^{01}$ by noting that the decomposition of

$$\begin{pmatrix} \mathbf{A}_{s_j} & \mathbf{B}_{s_j} \\ \mathbf{C}_{s_j} & \mathbf{T}_{s_j} \end{pmatrix} \quad \begin{pmatrix} \mathbf{A}_{s_j} & 0 \\ \mathbf{C}_{s_j} & \mathbf{T}_{s_j} \end{pmatrix} \begin{pmatrix} \mathbf{A}_{s_j} & \mathbf{B}_{s_j} \\ 0 & \mathbf{T}_{s_j} \end{pmatrix} \tag{3.25}$$

implies that

$$\mathbf{R}_{s_j} \quad \mathbf{T}_{s_j} \; 0 \; \mathbf{C}_{s_j} \mathbf{B}_{s_j}. \tag{3.26}$$

In the one-dimensional case, if $\mathbf{A}_{s_j}$ is banded with bandwidth m, then its LU factors will also be banded with bandwidth m, and thus they may be computed in $O(Nm^2)$. If $\mathbf{B}_{s_j}$ is also banded with this same bandwidth, then we may solve for $\mathbf{B}_{s_j}$ in $O(Nm^2)$; likewise for $\mathbf{C}_{s_j}$. For fixed relative accuracy ε (and hence fixed bandwidth m) this leads directly to the $O(N)$ procedure for computing $\mathbf{R}_{s_j}$ via the sparse incomplete block LU decomposition given by (3.26).

The two-dimensional case is more complicated. Each of the blocks on the left-hand side of (3.26) will in general exhibit a multibanded structure. Thus, one may expect the LU factors of $\mathbf{A}_{s_j}$ to fill in between the bands. Indeed, this is the case, but the fill-in which occurs is observed in practice to be fill-in with rapid decay, so that truncating to a given accuracy as we compute the LU factors results in a fast method for computing the reduction (as in the one-dimensional case).

There are many details involved in the description of the multiresolution LU decomposition, and we refer to [12] for a full treatment of them. We note finally that due to this algorithm the reduction procedure requires $O(N)$ operations.

3.4. Eigenvalues and Eigenvectors of the Reduced Operators

In this section we further investigate the relations between the spectra of the operators $\mathbf{S}_j$ and $\mathbf{R}_{s_j}$. In Section 3.1 we established relations between the spectral bounds of these operators and in this section we consider relations between the small eigenvalues and corresponding eigenvectors of the operators $\mathbf{S}_j$ and $\mathbf{R}_{s_j}$.

We will consider self-adjoint elliptic operators with compact inverses; this class includes variable-coefficient elliptic operators. For such an operator $\mathbf{S}$, the spectrum iidith0(sresptions).745 1 24eraltNm

$$0 \quad l_0 \; ¡ \; l_1 \; ¡ \; l_2 \; r \, r \, r.$$

The eigenvectors of such operators form an orthonormal basis in the Hilbert space H, and each eigenspace is a finite-dimensional subspace. Heuristically, e.g., in numerical literature, it is always assumed for elliptic operators that the eigenvectors which correspond to small eigenvalues are less oscillatory than those which correspond to large eigenvalues and the number of oscillations increases as $l_n \; r \; `$. For example, such statements typically form the basis for the heuristic justification of multigrid methods. There are many other examples of theorems where this property is a subject of consideration; see, e.g., [14]. Let us formulate a simple, general proposition capturing this property for the purposes of this paper.

DEFINITION. Let S be a subspace of the Hilbert space H. We will say that the

$$\mathbf{S}_j x \quad 1 x, \tag{3.27}$$

and $\mathbf{Q}_{j/1} x \quad 0$ implies that $\mathbf{P}_{j/1} x \quad x$, so that

$$\mathbf{T}_{\mathbf{S}_j} x \quad \mathbf{P}_j$$

LEMMA 3.1. *For a normal matrix* $\mathbf{M}$, *if*

$$\mathbf{M}x = \lambda x / \jmath, \tag{3.33}$$

then there exists an eigenvalue $\lambda_{\mathbf{M}}$ *of* $\mathbf{M}$ *such that*

$$\lambda = \lambda_{\mathbf{M}} \leq \frac{|\jmath|}{|x|}. \tag{3.34}$$

The proof of this lemma is straightforward. Let $\mathbf{G} = \mathbf{M} - \lambda \mathbf{I}$; then there is a singular value σ_0 of $\mathbf{G}$ such that

$$\sigma_0 = \inf_{|y| \neq 0} (\mathbf{G}^*\mathbf{G}y, y)$$

$$\mathbf{R}_{S_j} s \quad 1 s, \tag{3.43}$$

and

$$\mathbf{R}_{S_j} s \quad 1 (\mathbf{I} \ / \ \mathbf{B}_{S_j}^{*} \mathbf{A}_{S_j}^{0 2} \mathbf{B}_{S_j}) s. \tag{3.44}$$

The last equation gives rise to what we call the modified reduction procedure. As in (3.43), we would like to iterate the modified reduction procedure over many scales for (3.44) so that this form is preserved. To this end, we factor the operator $\mathbf{I} \ / \ \mathbf{B}_{S_j}^{*} \mathbf{A}_{S_j}^{0 2} \mathbf{B}_{S_j}$ by using the Cholesky decomposition and obtain

$$\mathbf{I} \ / \ \mathbf{B}_{S_j}^{*} \mathbf{A}_{S_j}^{0 2} \mathbf{B}_{S_j} \quad \mathbf{L}_{S_j} \mathbf{L}_{S_j}^{*}. \tag{3.45}$$

We rewrite (3.44) as

$$\mathbf{L}_{S_j}^{0 1} \mathbf{R}_{S_j} (\mathbf{L}$$

For the term neglected in (3.44), we follow the above considerations for the modified reduction procedure. After multiplying by $\mathbf{L}_{s_j}^{01}$ on the left and substituting (3.47) in (3.41), we have

$$\mathbf{Y}_{s_j} z \quad lz \;/\; l^2 \mathbf{L}_{s_j}^{01} \mathbf{B}_{s_j}^{*} \mathbf{A}_{s_j}^{02} d, \tag{3.51}$$

to which Lemma 3.1 may be applied. Let $\mathbf{Z}_{s_j} \quad \mathbf{I} \;/\; \mathbf{B}_{s_j}^{*} \mathbf{A}_{s_j}^{02} \mathbf{B}_{s_j}$. The lower spectral bound of $\mathbf{Z}_{s_j}$ is clearly bounded below by one (since $\mathbf{B}_{s_j}^{*} \mathbf{A}_{s_j}^{02} \mathbf{B}_{s_j}$ is positive-definite), and therefore \

TABLE 1

Condition Numbers and Lower Bounds for the A-Block of the Operator $0, r(a$

LU decomposition with working precision $\varepsilon \| h_j \|$. For a given accuracy δ, the bandwidth m of matrices which satisfy (3.13) (or its two-dimensional analogue) is given by $m \sim \delta^{-1/M}$, where M is the number of vanishing moments of the wavelet basis (see, e.g., [3] for details). Thus, as h_j decreases (and the scale becomes finer) it is necessary to keep a wider band in the LU decomposition. This thickening of the band as the scale becomes finer means that for the purposes of eigenvalue computations with fixed absolute accuracy, the reduction procedure is $O(N^{1+4/M})$ rather than $O(N)$.

This estimate is obtained if we choose the number of vanishing moments M based on the desired accuracy ε. A typical choice is $M \sim -\log(\varepsilon)$. With this choice we have the bandwidth $m \sim (\varepsilon$

We may therefore look for solutions of the equation

$$\left(\frac{\partial}{\partial t}\right)^2 u^j(x,t) \;/\; \mathbf{S}_j u^j(x$$

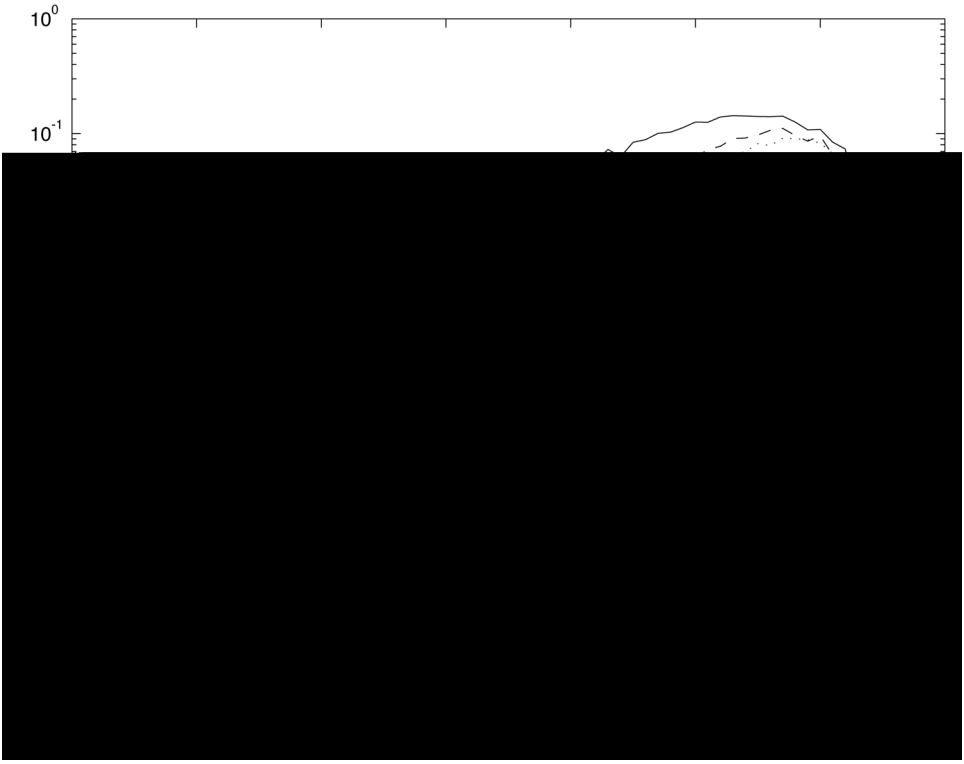

FIG. 2. Relative error of eigenvalues of the one-dimensional example operator reduced over four

FIG. 3. Relative error of eigenvalues of the one-dimensional example operator reduced via the modified reduction procedure over four scales, using wavelets with 4, 8, and 12 vanishing moments.

In Figs. 2 and 3 we perform reduction over four scales so that the reduced matrix is of size 64 1 64 (the original matrix is of size 1024 1 1024) and compare the 64 smallest eigenvalues of the original matrix with eigenvalues of the reduced 64 1 64 matrix. The three curves correspond to using compactly supported wavelets with different numbers of vanishing moments. Figure 2 was obtained by using the reduced operator $\mathbf{R}_{s_j}$ after four steps of reduction. Figure 3 demonstrates the performance of the modified reduction procedure. For some regimes of the spectrum, we observe that, as expected, increasing the number of vanishing moments increases the accuracy of the approximation.

Our second example illustrates some preliminary two-dimensional results. We consider the operator **S** 0 , r($a(x, y)$,) on the unit square with periodic boundary conditions; we define $a(x, y)$ 2 / cos(32p

FIG. 4. Relative error of eigenvalues of the operator 0 , r(2 /

large, require special treatment since their eigenvectors have large derivatives in the neighborhood where the above ratio is large. Since such operators represent interesting physical phenomena in elasticity, this also presents an interesting problem for the future.

Finally, the remarks in Section 3.5 appear to open a number of opportunities for reduction and homogenization of hyperbolic and parabolic problems. These problems present a separate subject matter with many practical applications.

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