On Generalized Gaussian Quadratures
for Exponentials and Their Applications

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We introduce new families of Gaussian-type quadratures for weighted integrals of exponential functions and consider their application to integration and interpolation of bandlimited functions.

We use a generalization of a representation theorem due to Carathéodory to derive the quadratures. For each positive measure, the quadrature are parameterized by eigenvalues of the Toeplitz matrix constructed from the trigonometric moments of the measure. For a given accuracy

the fo m

$$c_k = \sum_{j=1}^{M} {}_j e^{i\,{}_j k},\qquad(1.1)$$

fo $k = 1, 2, \ldots, N$ and M N, he e $-1 <$ $_j$ 1 and $_j > 0$. Ca ath odo ep e entation (1.1) ha been the fo ndation fo a n mbe of algo ithm fo pect al e timation; in pa tic la , [20] i kno n in elect ical enginee ing lite at e a the Pi a enko method. In thi pape e de elop a fa t algo ithm fo nding M, the pha e T015

A a method fo con t cting gene ali ed Ga ian q ad at e , o e lt a e limited to integ al (ith a fai l a bit a mea e) in ol ing ex ponential . O algo ithm in ol e nding eigen al e and eigen ecto of a Toeplit mat ix con t cted f om t igonomet ic moment of the mea e and then comp ting the oot on the nit ci cle fo app op iate eigenpol nomial . In pa tic la , each eigenpol nomial ith di tinct oot gi e i e to an identit hich, fo mall eigen al e , p o ide ith a Ga ian-t pe q ad at e and al o ith a ep e entation of po iti e de nite He mitian Toeplit mat ice . In the e identitie the i e of the eigen al e dete mine the acc ac of the q ad at e fo m la.

It t n o t that in the ca e of the eight leading to PSWF, the node of the co e ponding Ga ian q ad at e a e o (app op iatel caled to the inte al $[-1, 1]$) of di c ete PSWF co e ponding to mall eigen al e .

A an application, e e the ne q ad at.7(I)-4.9(n)-199.2gTj/Fx 7ng2(cted)cTj4o2a . 4.7(

The pape i o gani ed a follo . We p e ent a b ief de c iption of the Pi a enko method to obtain the cla ical Ca ath odo ep e entation and e de i e the e timate (1.2) in Section 2. In Section 3 e di c gene ali ed Ga ian q ad at e fo eighted integ al and p o e ome of thei p ope tie fo eight ppo ted in ide $[-1/2, 1/2]$. In Section 4 e int od ce ne familie of Ga ian-t pe q ad at e . We de elop a fa t algo ithm in Section 5 to comp te the node and eight of the e q ad at e . We ol e the app oximation p oblem (1.3) (1.5) in Section 6 and e it in the ne t t o ection to obtain q ad at e and inte polating ba e fo bandlimited f nction . We al o di c a io example to ill t ate the e e lt . Finall , concl ion a e p e ented in Section 9.

2. CARATHÉODORY REPRESENTATION

Ca ath odo ep e entation ol e the t igonomet ic moment p oblem and can be tated a follo (ee [8, Chap. 4]).

THEOREM 2.1. *Given N complex numbers* $c = (c_1, c_2, \ldots, c_N)$, *not all zero, there exist unique M* N, *positive numbers* $\rho = (_1, _2, \ldots, _M)$, *and distinct real numbers* $_1, _2, \ldots, _M$

2.1. Algorithm I: Method to Obtain M, θ, and ρ

(1) Given $c = (c_1, c_2, \ldots, c_N)$, we extend the definition of c_k to negative k as $c_{-k} = \overline{c_k}$ and we define c_0 so that the $(N + 1) \times (N + 1)$ Toeplitz matrix T_N of elements $(T_N)_{kj} = c_{j-k}$, has nonnegative eigenvalues and at least one eigenvalue is equal to zero.

(2) Define M as the rank of T_N. By construction, we have $M \leq N$. We also say that M is the *rank* of the representation (2.1).

(3) Let T_M be the top left principal submatrix of order $M + 1$ of T_N. That is, the matrix T_M has elements $(c_{j-k})_{0 \leq k,j \leq M}$. Find the eigenvector q corresponding to the zero eigenvalue of T_M.

(4) Construct the polynomial (eigenpolynomial) whose coefficients are the entries of the eigenvector q. As shown in [8, p. 58], the M roots of this eigenpolynomial are distinct and have absolute value 1. The phase of these roots are the numbers θ_j.

(5) Find the weights ρ by solving the Vandermonde system (2.1) for $k = 1, \ldots, M$. These will, in addition, satisfy $\sum_k \rho_k = c_0$.

Remark 2.1. With the extension of the sequence c_k, (2.1) is valid for $|k| \leq N$. If $q = (q_0, \ldots, q_M)$ is the eigenvector obtained in part (3) of Algorithm 2.1, then

$$\sum_{k=0}^{M} c_{k+s} q_k = 0, \tag{2.2}$$

for all s, $-N \leq s \leq 0$. In other words, we have found an order-M recurrence relation for the original sequence $\{c_k\}_{k=1}^{N}$.

Remark 2.2. In practice, we are interested in using Caratheodory representation if M is small compared with N, or, more generally, if most weights are smaller than the accuracy sought. However, in such cases, T_N has a large (numerical) null subspace that cause severe numerical problems in determining c_0, the rank M, and the eigenvector q.

Nevertheless, if the sequence c is the trigonometric moments of an appropriate weight, we will be able to modify the previous method in order to obtain the phase θ_j in an efficient manner. In this setting, the phases and weights in Caratheodory representation can be thought of as the nodes and weights of a Gaussian-type quadrature or, for weighted integrals. Once the phases are obtained, Theorem 2.2 assures that the computation of the weights is a well-posed problem. In Section 5.2 we present a fast algorithm to obtain the weights by evaluating certain polynomials at the nodes $e^{i\theta_j}$.

Remark 2.3. Given an Hermitian Toeplitz matrix T, let us consider its smallest eigenvalue $\lambda^{(N)}$

3. GENERALIZED GAUSSIAN QUADRATURES FOR EXPONENTIALS

3.1. Preliminaries: Chebyshev Systems

In this section, we collect some definitions and results related to Chebyshev systems. We follow mostly Karlin and Studden [12] (see also [13]). Readers familiar with this topic may skip this section.

A family of $n + 1$ real-valued functions $u_0, \ldots, u_n$ defined on an interval $I = [a, b]$ is a *Chebyshev system* (T-system) if any nontrivial linear combination

$$u(t) = \sum_{j=0}^{n} c_j u_j(t) \tag{3.1}$$

has at most n zeros on the interval I. This property of a T-system can be viewed as a generalization of the same property for polynomials. Indeed, the family $\{1, t, t^2, \ldots, t^n\}$ provides the simplest example of a Chebyshev system.

Alternatively, a T-system over $[a, b]$ may be defined by the condition that the $n + 1$ order determinant is non vanishing,

$$\det \begin{array}{cccc} u_0(t_0) & u_0(t_1) & \cdots & u_0(t_n) \\ u_1(t_0) & u_1(t_1) & \cdots & u_1(t_n) \\ \cdots & \cdots & \cdots & \cdots \\ u_n(t_0) & u_n(t_1) & \cdots & u_n(t_n) \end{array} = 0, \tag{3.2}$$

whenever $a \le t_0 < t_1 < \cdots < t_n \le b$. Without loss of generality, the determinant can be assumed positive.

Let $u_0, \ldots, u_n$ be a T-system on the interval I. The moment space $\mathcal{M}_{n+1}$ with respect to $u_0, \ldots, u_n$ is defined as the set

$$\mathcal{M}_{n+1} =$$

THEOREM 3.5 [12, VI, Sec. 4]. *For the periodic T-system* (3.5)*, a point*

In this section, we start by using Carathéodory representation and Theorem 3.7 from the previous section, to construct two different Gaussian quadrature for integrals with weight w. These quadratures are exact for trigonometric polynomials of appropriate degree.

We then generalize these types of quadratures further and develop a new family of Gaussian-type quadratures. This family of quadrature formulas is parameterized by the eigenvalues of the Toeplitz matrix

$$T = \{t_{l-k}\}_{0 \le k, l \le N}. \tag{4.2}$$

Among these new quadrature formulas, only those corresponding to eigenvalues of small size are of practical interest. In fact, the size of the eigenvalue determines the error of the quadrature formula. To compute the weights and nodes of these quadratures, we develop a new algorithm which may be viewed as a (major) modification of Algorithm 2.1. The new algorithm is described in Section 5. The main results of this section are gathered in Theorem 4.1.

We start by using Theorem 3.7 to write

$$t_k = \sum_{j=1}^{N} \rho_j e^{i\pi\theta_j k} + \rho_0(-1)^k, \qquad \text{for } |k| \le N, \tag{4.3}$$

for unique positive weights ρ_j and phases θ_j in $(-1, 1)$. Then, for any $A(z) = \sum_{|k| \le N} a_k z^k$ in Λ_N, the space of Laurent polynomials of degree at most N, we have

$$\int_{-1}^{1} A(e^{i\pi\tau}) w(\tau) d\tau = \sum_{|k| \le N} a_k t_k = \sum_{j=1}^{N} \rho_j A(e^{i\pi\theta_j}) + \rho_0 A(-1), \tag{4.4}$$

for unique positive weights ρ_j and node $e^{i\pi\theta_j}$.

Alternatively, using Carathéodory representation (2.1) applied to the sequence $c_k = t_k$, $1 \le k \le N$,

$$\int_{-1}^{1} A(e^{i\pi\tau}) w(\tau) d\tau = \sum_{j=1}^{M} \rho_j A(e^{i\pi\theta_j}) + (t_0 - c_0)\frac{1}{2}\int_{-1}^{1} A(e^{i\pi\tau}) d\tau$$

$$= \sum_{j=1}^{M} \rho_j A(e^{i\pi\theta_j}) + \lambda^{(N)}\frac{1}{2}\int_{-1}^{1} A(e^{i\pi\tau}) d\tau, \tag{4.5}$$

where $c_0 = \sum_{j=1}^{M} \rho_j$ and $\{e^{i\pi\theta_j}\}$ are the roots of the eigenpolynomial corresponding to the smallest eigenvalue $\lambda^{(N)}$ of T.

Note that (4.5) is again valid for all $A(z)$ in Λ_N and that the positive weights ρ_j and phases θ_j in $(-1, 1]$ are unique.

Thus, we have two different quadratures that may not coincide. However, by considering $w(\tau)$ supported inside $(-1/2, 1/2)$, (3.12) implies that w_0 in (4.4) decreases exponentially fast with N and, since $\min w(\tau) = 0$ for $|\tau| \le 1$, we have

$$\lim_{N} \lambda^{(N)} = 0, \tag{4.6}$$

4.2. Gaussian-Type Quadratures on the Unit Circle

In this section we present the main result of the paper. We derive new Gaussian-type quadratures valid for an eigenvalue of the matrix T rather than just the smallest eigenvalue λ_N. The new quadratures allow us to select the desired accuracy and thus, to construct accuracy-dependent families of quadratures.

The nodes of the quadratures in (4.5) are the roots of the eigenpolynomial corresponding to the least eigenvalue of T and, because of Carathéodory representation, we know that these roots are on the unit circle and that their weights are positive numbers. In our generalization, this standard property for the nodes and weights is no longer enforced. However, we will show that for nodes on the unit circle, the corresponding weights are real. Moreover, in all examples we have examined, for all small eigenvalues of T, their negative weights are associated with the nodes outside the support of the weight and are comparable in size with s. We believe this property to hold for a wide variety of weights.

We prove the following

THEOREM 4.1. *Assume that the eigenpolynomial $V^{(s)}(z)$ corresponding to the eigenvalue $\lambda^{(s)}$ of T has distinct, nonzero roots $\{\zeta_j\}_{j=1}^{N}$. Then there exist numbers $\{w_j\}_{j=1}^{N}$ such that*

(i) *For all Laurent polynomials $P(z)$ of degree at most N,*

$$\int_{-1}^{1} P(e^{i\pi t}) w(t)\,dt = \sum_{j=1}^{N} w_j P(\zeta_j) + \lambda^{(s)}\frac{1}{2}\int_{-1}^{1} P(e^{i\pi t})\,dt. \tag{4.12}$$

(ii) *For each root ζ_k with $|\zeta_k| = 1$, the corresponding weight w_k is a real number and*

$$w_k = \int_{-1}^{1} |L_k^s(e^{i\pi t})|^2 w(t)\,dt - \lambda^{(s)}\frac{1}{2}\int_{-1}^{1} |L_k^s(e^{i\pi t})|^2\,dt, \tag{4.13}$$

where

$$L_k^s(z) = \frac{V^{(s)}(z)}{(V^{(s)})'(\zeta_k)(z-\zeta_k)} \tag{4.14}$$

is the Lagrange polynomial associated with the root ζ_k.

(iii) *If $\lambda^{(s)}$ is a simple eigenvalue, then for $k = 1,\ldots,N$, the weight w_k is nonzero and*

$$\frac{1}{w_k} = \sum_{\substack{0\le l\le N \\ l\ne s}} \frac{V^{(l)}(\zeta_k)\, \overline{V^{(l)}}(\zeta_k)}{\lambda^{(l)} - \lambda^{(s)}}, \tag{4.15}$$

where $\overline{V^{(l)}}(z) = \overline{V^{(l)}}(z^{-1})$ is the reciprocal polynomial of $V^{(l)}(z)$.

In particular, for each ζ_k with $|\zeta_k| = 1$,

$$\frac{1}{w_k} = \sum_{\substack{0\le l\le N \\ l\ne s}} \frac{|V^{(l)}(\zeta_k)|^2}{\lambda^{(l)} - \lambda^{(s)}}. \tag{4.16}$$

(iv) *If $\lambda^{(s)}$ is a simple eigenvalue and all roots ζ_k are on the unit circle, then the set $\{w_k\}_{k=1}^{N}$ contains exactly s positive numbers and $N - s$ negative numbers.*

In particular, if $s = 0$ or $s = N$, then all w_k are negative or positive, respectively.

Remark 4.1. Our approach to obtain Gaussian quadrature does not use Szegő polynomials and is therefore substantially different than the one in [11]. We briefly explain the approach in [11]. Note that (4.9) and (4.10) show that the polynomials $\{V^{(k)}(z)\}$ are orthogonal with respect to both the usual inner product for trigonometric polynomials and the weighted inner product with weight $w(t)$. We can also construct Szegő polynomials $\{p_k(z)\}$ orthogonal with respect to $w(t)$ and such that each $p_k(z)$ has precise degree k [26]. For any k, the roots of $p_k(z)$ are all in $|z| < 1$ [8].

Szegő polynomials and their reciprocal induce para-orthogonal polynomials [11],

$$
p \qquad e \qquad r \qquad {}_a B_{nB}
$$

Fo k o t ide the   ppo t of the mea   e, e ha e ob e ed (Fig . 2, 3, and 5 8) that

$$\sum_{l: \ (s) > \ (l)} |V^{(l)}(\ _k)|^2$$

i a con tant of mode ate i e.

Th , the econd te m in (4.17) i $O(1/ \ ^{(s)})$ and the eight i indeed negati e and
o ghl of the i e of the eigen al e.

Remark 4.5. Fo the eight ith al e 1 in $(-1/2, 1/2)$ and 0 othe i e, the
eigenpol nomial a e the di c ete PSWF. Fo the e f nction, e kno that all eigen al e
a e imple and that all eigenpol nomial oot a e on the nit ci cle [23].

COROLLARY 4.1. *Under the assumptions of Theorem* 4.1, *it follows that the Toeplitz
matrix* $\boldsymbol{T}$ *in (*4.2*) has the following representation as a sum of rank-1 Toeplitz matrices,*

$$(\boldsymbol{T} - \ ^{(s)}\boldsymbol{I})_{kl} = \sum_{j=1}^{N} w_j \ _j^{l-k},$$

where $\ ^{(s)}$, w_j, *and* $\ _j$ *are as in (4.12).*

Thi co olla   ho ld be compa ed ith Rema k 2.3 noting that, in the co olla , $\ ^{(s)}$ i
not nece a il the lea t eigen al e of $\boldsymbol{T}$. Fo an alte nati e de i ation ee [4].

Proof of Theorem 4.1. (1) Fo $\boldsymbol{x} = (x_0, \dots, x_N)$ $\mathbb{C}^{N+1}$, let   de ne

$$\sum_{l=-L}^{L} x_{l+L} z^l, \qquad \text{if } N = 2B$$

$$A_{\boldsymbol{x}}(z) =$$

$<$

(3) Let P N; then $z^N P(z)$ i a pol nomial of at mo t deg ee $2N$, and ince $z^L V^{(s)}(z)$
i a pol nomial of deg ee N, b E clidean di i ion, the e ex.i t pol nomial $q(z)$ and $r(z)$
of deg ee at mo t N and $N - 1$ ch that

$$z^N P(z) = z^L V^{(s)}(z) q(z) + r(z).$$

Th ,

$$P(z) = V^{(s)}(z) Q(z) + R(z), \tag{4.19}$$

he e $Q(z)$ L and $R(z)$ ha the fo m $R(z) = \sum_{k=1}^{N} r_k z^{-k}$ and hence

$$\int_{-1}^{1} R(e^{i \ t}) \, dt = 0.$$

U ing the fact that $\{V^{(l)}\}_{l=0}^{N}$ i a ba i of L, e   ite

$$\overline{Q(e^{i \ t})} = \sum_{l=0}^{N} d_l V^{(l)}(e^{i \ t}),$$

he e d_l a e ome comple coef cient .

U ing (4.10) and (4.18), e m ltipl both ide of (4.19) b $w(t)$ and integ ate to obtain

$$\int_{-1}^{1} P(e^{i \ t}) w(t) \, dt = \qquad^{N}$$

and thus, considering $k = j$, (4.15) follows. Note that we need $^{(s)}$ to be simple to guarantee $^{(l)} - ^{(s)} = 0, l = s$ in (4.20).

If we fix the left hand side of (4.20) as the entries A_{kj} of a matrix A and let B be the matrix of entries

$$B_{lk} = V^{(l)}(z_k), \qquad \text{where } 0 \le l \le N, l = s, \text{ and } 1 \le k \le N, \qquad (4.21)$$

we can prove (4.20) by showing that $BA = B$ and that B is nonsingular.

For the latter claim, we simply check that the columns of B are linearly independent. Indeed, let $a_l,\ l = s$, be constants such that

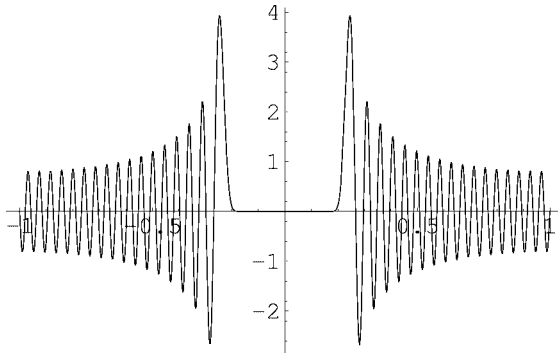$$\sum_{l=s} a_l V^{(l)}(z_k) = 0, \qquad \text{for } k = 1, \ldots, N.$$

It follows that the polynomial $P(z) = \sum_{l=s} a_l V^{(l)}(z)$ $_L$ has the $N = 2L$ distinct roots z_k. Since P and $V^{(s)}$ have the same degree and the same N distinct roots, $P(z) = cV^{(s)}(z)$, for some constant c. By (4.9), $V^{(s)}(z)$ is orthogonal to all the other eigenpolynomials and so $a_l = 0$.

To show that $BA = B$, we substitute $P(z) = V^{(l)}(z)V^{(m)}(z)$ in (4.12) to obtain

$$\int_{-1}^{1} V^{(l)}(e^{i \cdot t})$$

FIG. 2. Modi ed eigenpol nomial $e^{-i\,t(N/2)}V^{(30)}(e^{i\,t})$ on the inte al $[-1, 1]$, he e $N = 97$ and $V^{(30)}(e^{i\,t})$ i the eigenpol nomial co e ponding to the eigen al e $^{(30)}$ in Example 1. The pha e facto $e^{-i\,tN/2}$ i int od ced to make thi f nction eal.

EXAMPLE 1. Fi t e con ide the eight

$$w(t) = \begin{cases} 1, & t \quad [-a, a], \ a \quad 1/2, \\ 0, & \text{el e he e.} \end{cases} \tag{4.24}$$

Fo thi eight, the eigenpol nomial $V^{(l)}(e^{i\,t})$ of the $N + 1 \times N + 1$ Toeplit mat x T a e the di c ete PSWF [23]. Th the eigenpol nomial $V^{(l)}(e^{i\,t})$ ha all of it e o on the nit ci cle. Mo eo e , it ha exactl l e o fo t in the inte al $(-a, a)$ and N e o fo t in $[-1, 1]$. In thi example e ha e elected $N = 97$, $a = 1/6$, $c = 15$. We then con t ct the mat x T and comp te the eigenpol nomial co e ponding to the eigen al e

$$^{(30)} = 9.77306136381891632828 \cdot 10^{-16}. \tag{4.25}$$

The eigenpol nomial $V^{(30)}(e^{i\,t})$ i ho n in Fig . 2 and 3. Location of the e o on the nit ci cle a e di pla ed in Fig. 4. We then e the q ad at e fo m la co e ponding to thi eigen al e and tab late the eight in Table I. Note that the eight fo node in ide the inte al $[-1/6, 1/6]$
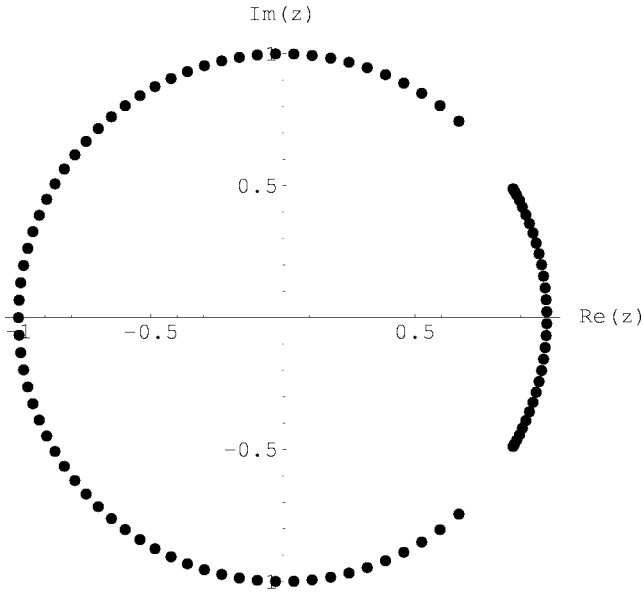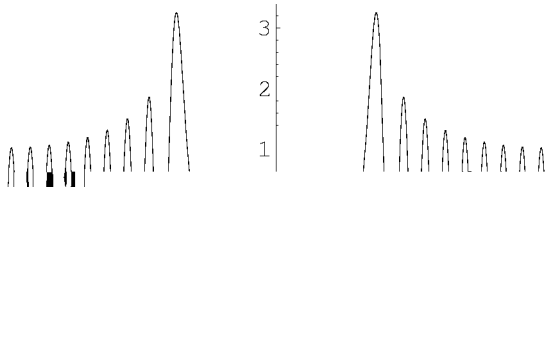
FIG. 4. Location of the e o on the nit ci cle fo the eigenpol nomial $V^{(30)}$ in Ex ample 1.

TABLE I
Table of Weights for the Quadrature Formula with $\lambda^{(30)}$ in Example 1

#	Weight	#	Weight
1	$-1.0328 \cdot 10^{-17}$	50	0.04437549133235668283
2	$-1.0328 \cdot 10^{-17}$	51	0.04419611220330997984
3	$-1.0329 \cdot 10^{-17}$	52	0.04382960375644760677
$\vdots$	$\vdots$	53	0.04325984471286061543
		54	0.04246105337417774134
33	$-1.3518 \cdot 10^{-17}$	55	0.04139574827622469674
34	$-1.6030 \cdot 10^{-17}$	56	0.04001188663952018400
35	0.00580295532842819966	57	0.03823923547752508920
36	0.01310603337477264417	58	0.03598544514201341779
37	0.01959211245475268191	59	0.03313334531810570720
38	0.02506789313597245367	60	0.02954323947353217723
39	0.02954323947353217723	61	0.02506789313597245367
40	0.03313334531810570720	62	0.01959211245475268191
41	0.03598544514201341779	63	0.01310603337477264417
42	0.03823923547752508920	64	0.00580295532842819966
43	0.04001188663952018400	65	$-1.6030 \cdot 10^{-17}$
44	0.04139574827622469674	66	$-1.3518 \cdot 10^{-17}$
45	0.04246105337417774134		
46	0.04325984471286061543		
47	0.04382960375644760677		
48	0.04419611220330997984		
49	0.04437549133235668283		

FIG. 5. Modi ed eigenpol nomial (ee Fig. 2) on the inte al $[-1, 1]$ co e ponding to the eigen al e [28] in Ex ample 2.

EXAMPLE 2. We con ide the eight

$$w(t) = \begin{cases} |t|/a, & t \in [-a, a], \ a \ 1/2, \\ 0, & el \ e \ he \ e. \end{cases} \quad (4.26)$$

In thi ex ample e ha e elected $N = 61$, $a = 1/4$, $c = 15$. We then con t ct the mat ix T and comp te the eigenpol nomial co e ponding to the eigen al e

$${}^{(28)} = 1.11598931688523706280 \cdot 10^{-14}. \quad (4.27)$$

The eigenpol nomial $V^{(28)}(e^{i \ t})$ i ho n in Fig . 5 and 6.

EXAMPLE 3. We con ide a non mmet ic eight

$$w(t) = \begin{cases} 1 + t/a, & t \in [-a, a], \ a \ 1/2, \\ 0, & el \ e \ he \ e. \end{cases} \quad (4.28)$$

FIG. 6. The ame f nction of Fig. 5 on the inte al $[-1/4, 1/4]$.

FIG. 7. Modified eigenpolynomial (see Fig. 2) on the interval $[-1, 1]$ corresponding to the eigenvalue $^{(28)}$ in Example 3.
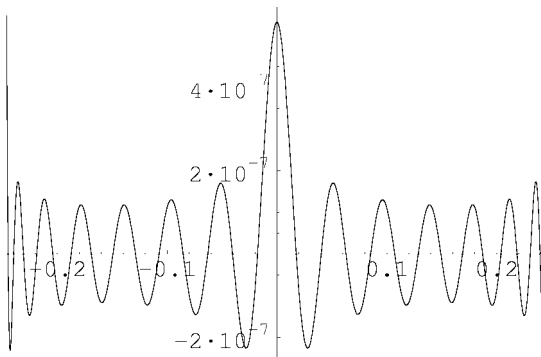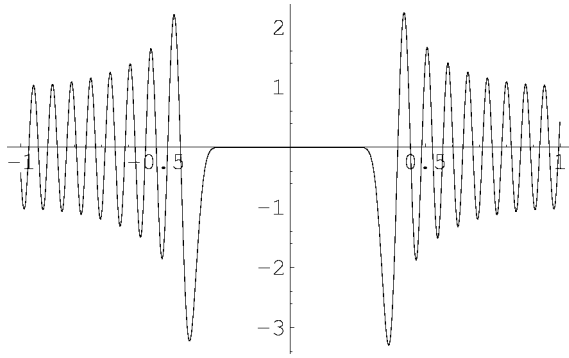
In this example we have selected $N = 61$, $a = 1/4$, $c = 15$. We then construct the matrix T and compute the eigenpolynomial corresponding to the eigenvalue

$$^{(28)} = 4.68165338379692121389 \cdot 10^{-15}. \tag{4.29}$$

The eigenpolynomial $V^{(28)}(e^{i\,t})$ is shown in Figs. 7 and 8. Although we do not have a proof at the moment, it appears that there is a class of weights for which eigenpolynomials corresponding to small eigenvalues mimic the behavior of the discrete PSWF with respect to location of zeros. In Example 3 we know that all zeros are on the unit circle due to Theorems 4.2 and 4.3.

In Table II we illustrate the performance of quadrature for different bandlimits c. This table should be compared with [29, Table 1]. The performance of both sets of quadrature is the similar. Yet the two quadratures are quite different as can be seen by comparing Table III with [29, Table 5]. Although the accuracy is almost identical, approximately 10

TABLE II
Quadrature Performance for Varying Bandlimits

c	# of node	Ma𝗑 im. m e o
20	13	$1.2 \cdot 10^{-7}$
50	24	$1.1 \cdot 10^{-7}$
100	41	$1.6 \cdot 10^{-7}$
200	74	$1.8 \cdot 10^{-7}$
500	171	$1.4 \cdot 10^{-7}$
1000	331	$2.4 \cdot 10^{-7}$
2000	651	$1.2 \cdot 10^{-7}$
4000	1288	$3.7 \cdot 10^{-7}$

5. A NEW ALGORITHM FOR CARATHÉODORY REPRESENTATION

5.1. Algorithm 2

We no𝗑 de c ibe an algo ithm fo comp ting q ad at e ia a Ca ath odo -t pe app oach ba ed on Theo em 4.1. It i ea to ee that, altho gh the e a e imila itie 𝗑 ith

TABLE III
Quadrature Nodes for Exponentials with Maximum Bandlimit $c = 50$

Node	Weight
-0.99041609489889	2.42209284787E-02
-0.95238829377394	5.04152570050E-02
-0.89243677566550	6.82109308489E-02
-0.81807124037876	7.96841731718E-02
-0.73438712699465	8.71710040243E-02
-0.64454148960251	9.22000859355E-02
-0.55050369342444	9.56668891250E-02
-0.45355265507507	9.80920675810E-02
-0.35456254990620	9.97843340729E-02
-0.25416536256280	1.00930070892E-01
-0.15284664158549	1.01641529848E-01
-0.05100535080412	1.01982696564E-01
0.05100535080412	1.01982696564E-01
0.15284664158549	1.01641529848E-01
0.25416536256280	1.00930070892E-01
0.35456254990620	9.97843340729E-02
0.45355265507507	9.80920675810E-02
0.55050369342444	9.56668891250E-02
0.64454148960251	9.22000859355E-02
0.73438712699465	8.71710040243E-02
0.81807124037876	7.96841731718E-02
0.89243677566550	6.82109308489E-02
0.95238829377394	5.04152570050E-02
0.99041609489889	2.42209284787E-02

Pi a enko' method, the co e ponding algo ithm a e b tantiall diffe ent. We plan to
add e implication fo ignal p oce ing in a epa ate pape .

(1) Gi en t_k, the t igonomet ic moment of a mea e, e con t ct the $(N + 1) \times (N + 1)$ Toeplit mat ix T_N ith element $(T_N)_{kj} = t_{j-k}$. Thi mat ix i po iti e de nite and ha a la ge n mbe of mall eigen al e .

(2) Fo a gi en acc ac , e comp te the in e e of the Toeplit mat ix $T_N - I$. Fo a elf-adjoint Toeplit mat ix, it i f cient to ol e $(T_N - I p Po n$ $prj V$ Q p $ffeyr$

If we define

$$Q(z) = \prod_{k=1}^{M} (z - \zeta_k) = \sum_{k=0}^{M} q_k z^k, \tag{5.2}$$

then, for any polynomial P of degree at most $M - 1$,

$$\frac{P(z)}{Q(z)} = \sum_{r=1}^{M} \frac{P(\zeta_r)}{Q'(\zeta_r)(z - \zeta_r)}.$$

Then, for $|z| < \min|\zeta_r|^{-1}$,

$$\frac{z^{M-1}}{z^{M}} \frac{P(z^{-1})}{Q(z^{-1})} = \sum_{r=1}^{M} \frac{P(\zeta_r)}{Q'(\zeta_r)} \sum_{k=0}^{+\infty} \zeta_r^k z^k = \sum_{k=0}^{+\infty} \sum_{r=1}^{M} \frac{P(\zeta_r)}{Q'(\zeta_r)} \zeta_r^k z^k. \tag{5.3}$$

Now choose P to be the unique polynomial with $P(\zeta_r) = \zeta_r Q$

Thi algo ithm i eq i alent to the follo ing facto i ation of the in e e of the Vande monde mat i in te m of a diagonal mat i , it t an po e V^t, and a t iang la Hankel mat i ,

$$
V^{-1} = \begin{pmatrix} \dfrac{1}{Q(\ _1)} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & \dfrac{1}{Q(\ _M)} \end{pmatrix} \quad V^t \begin{pmatrix} q_1 & q_2 & \cdots & q_M \\ q_2 & \cdots & q_M & 0 \\ \vdots & & & \vdots \\ & & \cdots & 0 \\ q_M & \cdots & 0 & 0 \end{pmatrix}. \tag{5.5}
$$

Thi de c iption i a pa tic la ca e of the in e ion fo m lae fo L ne Vande - monde [21] o clo e to Vande monde mat ice [9, Co olla 2.1, p. 157]. We can tate tho e e lt a (ee [21, p. 548])

$$
V^{-1} = \begin{pmatrix} x_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & x_M \end{pmatrix} \quad V^t \begin{pmatrix} & -y_2 & -y_3 & \cdots & 1 \\ & -y_3 & \cdots & 1 & 0 \\ & \vdots & & & \vdots \\ & & & \cdots & 0 \\ & 1 & \cdots & 0 & 0 \end{pmatrix},
$$

 he e the ecto $x = (x_1, \ldots, x_M)^t$ and $y = (y_1, \ldots, y_M)^t$ a e ol tion of

$$
Vx = (0, \ldots, 1)^t \qquad \text{and} \qquad V^t y = [\ _r^M]_{r=1}^M.
$$

Since $_r$ a e the oot of $Q(z)$, e can take $y = -(q_0, \ldots, q_{M-1})^t$, and if $B(z) = z^M$ in (5.4), then $P(z) = 1$ and $x = (1/Q(\ _1), \ldots, 1/Q(\ _M))^t$.

Remark 5.1. Fo Algo ithm 5.1, e t obtained the eigen ecto q co e ponding to an eigen al e clo e to . Th , tep (1) of the Vande monde algo ithm i al ead accompli hed and tep (2) can be pe fo med ing the FFT. F the mo e, the node $_k$ belong to the nit ci cle and, ia the neq all   paced fa t Fo ie t an fo m, e ha e a fa t algo ithm to obtain the eight .

Remark 5.2. A an ex ample, e e thi app oach to de i e the ol tion of the Vande monde   tem ith node at $_r = e^{i2\ (r-1)/M}$, $1\ \ r\ \ M$. In thi ca e, $Q(z) = 1 - \ _h$

Proof of Theorem 6.1. Let

$$u(y) = \int_{-1}^{1} \phi(t) e^{i\pi ty} \, dt,$$

and, for each m, define the spline of order $2m - 1$ interpolating $u(y)$ at the integers,

$$a(y) = \sum_{k} u(k) L_{2m-1}(y - k) = \int_{-1}^{1} \phi(t) S_{2m-1}(y, e^{i\pi t}) \, dt.$$

By (6.7),

$$|u(y) - a(y)| \le C_3 \int_{-}^{} \phi(t) |t|^{2m} \, dt \le C_3 \gamma^{2m} M_1,$$

where $M_1 = \int_{-1}^{1} \phi(t) \, dt$. We choose m such that $C_3 \gamma^{2m} M_1 < \epsilon/4$.

On the other hand, for each N, Theorem 3.7 allows us to represent the moments $u(k)$, $|k| \le N$,

$$u(k) = \int_{-1}^{1} \phi(t) e^{i\pi kt} \, dt = \sum_{j=1}^{N} w_j e^{i\pi \gamma_j k} + w_0 (-1)^k, \tag{6.9}$$

where

$$w_0 \le \frac{4M_1}{2 + (2 + \sqrt{3})^N + (2 - \sqrt{3})^N}. \tag{6.10}$$

Let

$$\tilde{u}(y) = \sum_{j=1}^{N} w_j e^{i\pi \gamma_j y};$$

then $u(k) = \tilde{u}(k) + w_0(-1)^k$ for $|k| \le N$, and defining

$$\tilde{a}(y) = \sum_{k} \tilde{u}(k) L_{2m-1}(y - k) = \sum_{j=1}^{N} w_j S_{2m-1}(y, e^{i\pi \gamma_j}),$$

(6.7) gives the estimate

$$|\tilde{u}(y) - \tilde{a}(y)| \le C_3 \sum_{j=1}^{N} w_j |\gamma_j|^{2m} \le C_3 \gamma^{2m} (u(0) - w_0) \le C_3 \gamma^{2m} M_1 < \frac{\epsilon}{4}.$$

We have shown that $u(y)$ is close to $a(y)$ and $\tilde{u}(y)$ is close to $\tilde{a}(y)$. To finish the proof, we need to show that $|a(y) - \tilde{a}(y)| < \epsilon/2$, for $|y| \le dN + 1$. Now,

$$a(y) - \tilde{a}(y) = \sum_{|k| \le N} w_0 (-1)^k L_{2m-1}(y - k) + \sum_{|k| > N} (u(k) - \tilde{u}(k)) L_{2m-1}(y - k)$$

$$= w_0 S_{2m-1}(y, e^{i\pi}) + \sum_{|k| > N} (u(k) - \tilde{u}(k) - w_0(-1)^k) L_{2m-1}(y - k)$$

and

$$|u(k) - \tilde{u}(k) - w_0 (-1)^k| \le |u(k)| + |\tilde{u}(k)| + w_0 \le \sum_{j=0}^{N} w_j + \sum_{j=1}^{N} w_j + w_0$$

$$\le 2u(0) = 2M_1,$$

where we used (6.9).

Since J_{2n} i an e en f nction, e ha e

$$v(x) = \int_{-1}^{1} \check{w}(\ )J_{2n}(cx\ )d\ .$$

(7.4)

U ing

$$J_{2n}(\ ) = \frac{(-1)^n}{}$$

where

$$\bar{v}_j = \sum_{k=1}^{M} w_k \ _j(\ _k), \tag{7.13}$$

and the nodes $_k$ and the weights w_k are the same as in (1.4).

Fo la ge c, the pect m of F_c can be di ided into th ee g o p . The t g o p contain app ox imatel $2c/$ eigen al e $_{\bar{v}}$ ith ab ol te al e e clo e to 1. The a e follo ed b o de log c eigen al e $_{\bar{v}}$ ho e ab ol te al e make an ex ponentiall fa t t an ition f om 1 to 0. The thi d g o p con i t of ex ponentiall deca ing eigen al e that a e e clo e to e o. Fo p eci e tatement ee [14, 24, 25, 29].

The efo e, it follo f om (7.12) that, fo the t 2c/ eigenf nction , the integ al in (7.11) a e $_{\bar{v}}$ ell app ox imated b the q ad at e in (7.13). To p o e (7.12), e (7.10), to $_{\bar{v}}$ ite

$$v_j - \bar{v}_j = \frac{1}{_j} \int_{-1}^{1} \ \left[\int_{-1}^{1} w(\ )e^{ic\ t}d\ - \sum_{k=1}^{M} w_k e^{ic\ _k t} \right] \ _j(t)\, dt. \tag{7.14}$$

Since $|t| \ 1,_{\bar{v}}$ e ha e

$$\left| \int_{-1}^{1} w(\ )e^{ic\ t}d\ - \sum_{k=1}^{M} w_k e^{ic\ _k t} \right| \ , \tag{7.15}$$

and $_j\ _2 = $ **anidn**plie TD0.0052 Tc(.14 41e012)Tj67 8Tj/(i)0.9(m Tm()8.6483 0 TD-94h5)T1 T

In con ide ing bandlimi ed f. nction $_{\mathbb{K}}$ e $_{\mathbb{K}}$ ill . e the PSWF (ee [15, 24], and a mo e ecent pa

$_c$ in (7.9) $_{\mathbb{K}}$ itheigen al

$_j \, \boldsymbol{j} =$

By setting

$$\lambda_l = w_l \sum_{j=0}^{M-1} \varphi_j (b/c) \, \psi_j (t_l),$$ (8.18)

and observing that $|\lambda_M|$ and that $|\lambda_j| \le |\lambda_M|$ for $j > M$, we obtain (8.5) and (8.6). ∎

We now construct two useful bases as linear combinations of the functions $\{e^{ict_l x}\}_{l=1}^M$. First, let us consider the following algebraic eigenvalue problem,

$$\sum_{l=1}^M w_l e^{ict_m t_l} \, \psi_j (t_l) = \gamma_j \, \psi_j (t_m),$$ (8.19)

where t_l and w_l are the same as in (8.1). By solving (8.19), we find γ_j and $\psi_j (t_l)$. We then consider functions $\psi_j, j = 1, \ldots, M$, defined for any x as

$$\psi_j (x) = \frac{1}{\gamma_j} \sum_{l=1}^M w_l e^{icxt_l} \, \psi_j (t_l).$$ (8.20)

The functions ψ_j in (8.20) are linear combinations of the exponentials $\{$ the

matrices, the exist an orthonomal basis of real eigenvectors. Thus, computed via (8.22), we assume q_l^j to be a real orthogonal matrix, and then

$$\sum_{j=1}^{M} \overline{w_l} \phi_j(t_l) \phi_j(t_m) \overline{w_m} = \delta_{lm} \tag{8.23}$$

and

$$\sum_{l=1}^{M} \phi_j(t_l) w_l \phi_j(t_l) = \delta_{jj}. \tag{8.24}$$

We have

$$\int_{-1}^{1} \phi_j(t) \phi_j(t) \, dt = \frac{1}{\delta_j \delta_j} \sum_{l,l=1}^{M} w_l w_l \phi_j(t_l) \phi_j(t_l) \int_{-1}^{1} e^{ict(t_l + t_l)} \, dt \tag{8.25}$$

and, from (8.1), we obtain

$$\left| \int_{-1}^{1} \phi_j(t) \phi_j(t) \, dt - \frac{1}{\delta_j \delta_j} \sum_{l,l=1}^{M} w_l w_l \phi_j(t_l) \phi_j(t_l) \sum_{k=1}^{M} w_k e^{ict_k(t_l + t_l)} \right|$$

$$\frac{\varepsilon^2 \sum_{k=1}^{M} w_k}{|\delta_j||\delta_j|}. \tag{8.26}$$

Let us now construct interpolating basis a linear combination of the exponential $\{e^{icxt_l}\}_{l=1}^{n}$. We define function R_k, $k = 1, \ldots, M$, a

$$R_k(x) = \sum_{l=1}^{M} r_{kl} e^{icxt_l}, \tag{8.27}$$

where

$$r_{kl} = \sum_{j=1}^{M} w_k \quad_j(t_k)\frac{1}{j} \quad_j(t_l)w_l = \sum_{j=1}^{M} \overline{w_k} q_k^j \frac{1}{j} q_l^j \overline{w_l}. \tag{8.28}$$

By direct evaluation in (8.19) and (8.23), we see if that function R_k are interpolating,

$$R_k(t_m) = \quad_{km}. \tag{8.29}$$

Let us show that the integration of $R_k(t)e^{iat}$, where $|a| \quad c$, yield a one-point quad at the le of accuracy $O(\ )$.

PROPOSITION 8.3.   *For* $|a| \quad c$, *let*

$$\quad_k = \int_{-1}^{1} R_k(t)e^{iat} \, dt - w_k e^{iat_k}. \tag{8.30}$$

Then we have

$$|\quad_k| \qquad_2 \qquad \overline{M} \frac{\max_{k=1,\ldots,M}|w_k|}{\min_{k=1,\ldots,M}|\quad_k|} \quad_2, \tag{8.31}$$

where $\quad_2 = \sqrt{\sum_{k=1}^{M}|\quad_k|^2}$.

Proof.   Using (8.27) and (8.29),

$$\sum_{l=1}^{M} r_{kl} \sum_{m=1}^{M} w_m e^{ict_m(t_l + a/c)} = \sum_{m=1}^{M} w_m R_k(t_m)e^{iat_m} = w_k e^{iat_k}, \tag{8.32}$$

and, therefore, $\quad_k$ in (8.30) can be written a a matrix-vector multiplication $\quad_k = \sum_{l=1}^{M} r_{kl}s_l$, where

$$s_l = \int_{-1}^{1} e^{ict(t_l + a/c)} \, dt - \sum_{m=1}^{M} w_m e^{ict_m(t_l + a/c)}. \tag{8.33}$$

The inequality (8.31) i then obtained via the al l^2-norm estimate, taking into account that the matrice q_k^j and q_l^j in (8.28) are orthogonal and that, for function e^{iax}, where $|a| \quad c$, (8.1) implie $|s_l| \quad^2$. ■

We have observed (via computation) that $\max_{k=1,\ldots,M}|w_k| = O(1)$ and $\min_{k=1,\ldots,M}|\quad_k| = O(\ )$ in (8.31), the e ulting in $\quad_2 = O(\ )$. Next we derive a weak e timate howing that the function R_k are close to being an interpolating basis for band-limited exponential.

PROPOSITION 8.4. *For every b, $|b| \le c$, let us consider the function*

$$\mathcal{E}_b(t) = e^{ibt} - \sum_{k=1}^{M} e^{ibt_k} R_k(t). \tag{8.34}$$

Then, for every $|a| \le c$, we have

$$\left| \int_{-1}^{1} \mathcal{E}_b(t) e^{iat} \, dt \right| \le \left(1 + M \frac{\max_{k=1,\ldots,M} |w_k|}{\min_{k=1,\ldots,M} |\delta_k|} \right) \delta^2. \tag{8.35}$$

Proof. Using (8.30), we have

$$\int_{-1}^{1} \mathcal{E}_b(t) e^{iat} \, dt = \int_{-1}^{1} e^{i(b+a)t} \, dt - \sum_{k=1}^{M} w_k e^{i(b+a)t_k} - \sum_{k=1}^{M} e^{ibt_k} \delta_k, \tag{8.36}$$

where

$$\delta_k = \int_{-1}^{1} R_k(t) e^{iat} \, dt - w_k e^{iat_k}. \tag{8.37}$$

Applying (8.1), we obtain

$$\left| \int_{-1}^{1} \mathcal{E}_b(t) e^{iat} \, dt \right| \le \delta^2 + \overline{M} \delta_2. \tag{8.38}$$

The estimate (8.35) then follows from Proposition 8.3. ∎

Remark 8.2. Using the functions R_k, $k = 1, \ldots, M$, on a hierarch of interval, it is possible to construct a multiresolution basis (for a finite number of scales) similar to multiwavelet bases. We will consider such construction and its applications elsewhere.

8.1. Examples

For the weight

$$\chi(t) = \begin{cases} 1, & t \in [-a, a], \ a \le 1/2, \\ 0, & \text{otherwise,} \end{cases} \tag{8.39}$$

we construct a 30-node quadrature formula so that (8.1) is satisfied with $\delta^2 \le 10^{-15}$. We
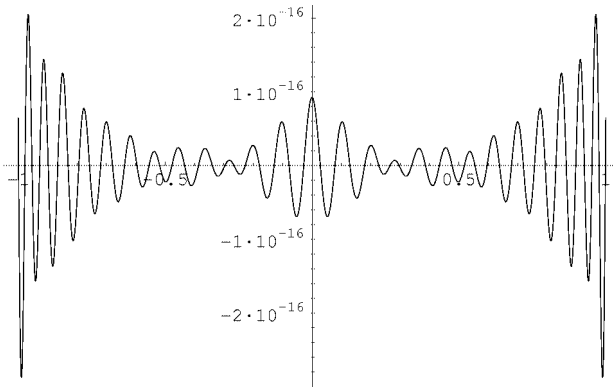
2

FIG. 9. E o in (8.1) fo Ex ample 1.

he e P_9 i the Legend e pol nomial of deg ee 9. The e th ee f nction a e not pe iodic and e e
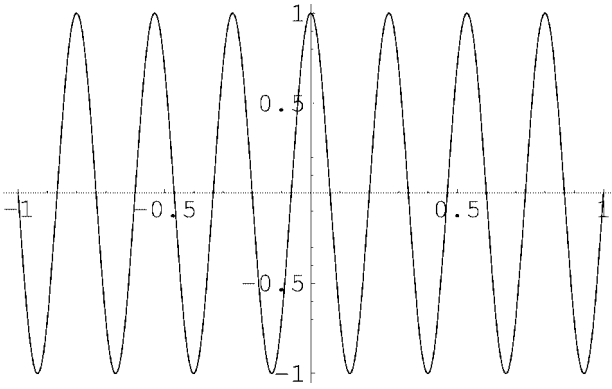
FIG. 11. Function $g_1(t)$ on the interval $[-1, 1]$.

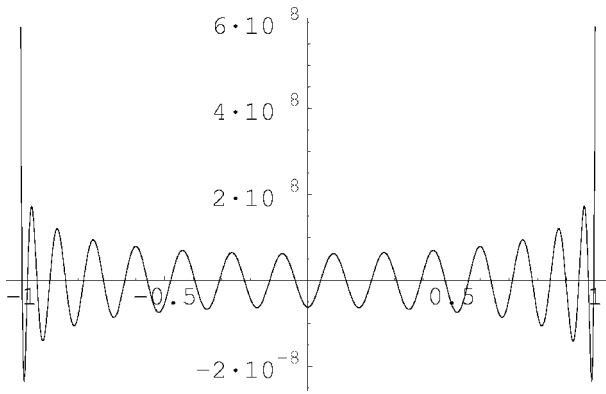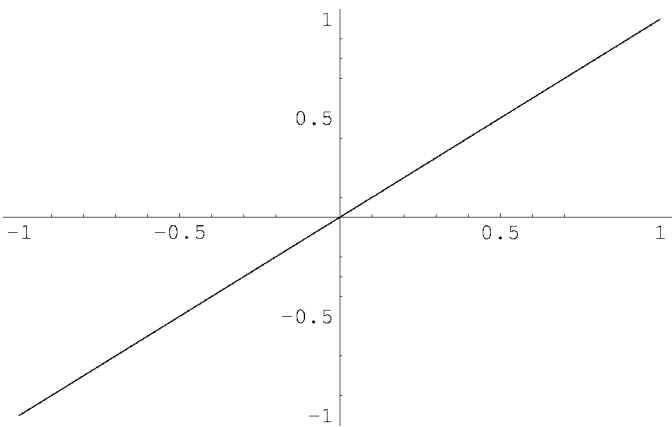**FIG. 12.** Difference $g_1(t) - \check{g}_1(t)$ on the interval $[-1, 1]$.

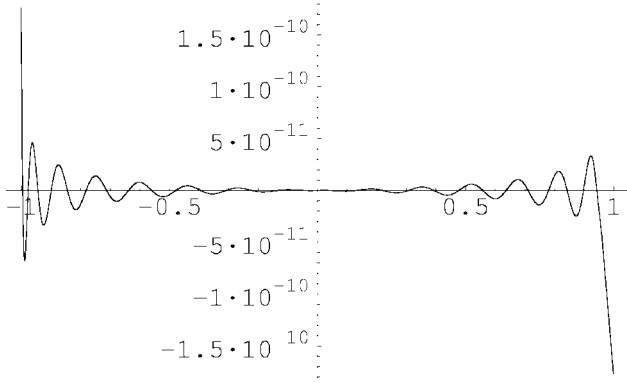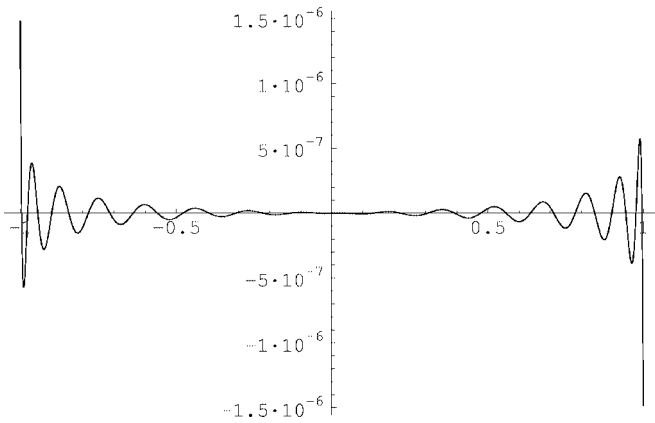**FIG. 13.** Function $g_2(t)$ on the interval $[-1, 1]$.

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FIG. 14. Diffe ence $g_2(t) - \bar{g}_2(t)$ on the inte al $[-1, 1]$.
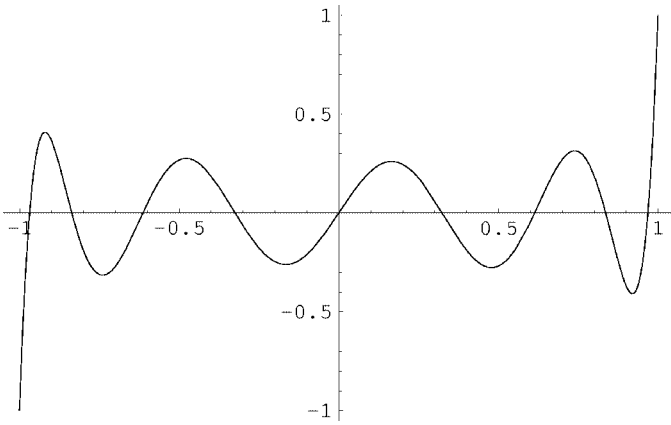
FIG. 15. F nction $g_3(t)$ on the inte al $[-1, 1]$.

FIG. 16. Diffe ence $g_3(t) - \bar{g}_3(t)$ on the inte al $[-1, 1]$.

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exponential decay (see Fig. 1). For small eigenvalues, the quadratures are of practical interest.

The remarkable feature of the quadratures is that they have nodes outside the support of the measure and, as it turns out, the corresponding weights are negative and small, roughly of the size of the eigenvalue. The case corresponding to the smallest eigenvalue is equivalent to the classical Gauss method of representation.

As an application of the new quadratures, we show how to approximate and integrate several (essentially) bandlimited functions. We also have constructed, using quadrature nodes and for a given precision, an interpolating basis for bandlimited functions on an interval.

In the paper we made a number of observations for which we do not have proofs. Let us finish by stating two unresolved issues. First, it is desirable to have tight uniform estimates for the L^∞-norm of the PSWF (with a fixed bandlimiting constant) or, ideally, for the eigenfunctions associated with more general weights. Second, we conjecture that in Theorem 4.1, it is not necessary to require distinct roots for the eigenpolynomial since it might be a consequence of the eigenvalue being simple. We have neither a proof nor a counterexample at this time.

APPENDIX: PROOF OF THEOREM 2.2

We use a technique that goes back to [2] (see [28, Theorem 7.3] and [19, Chapter 5] for more details), which involves the Fejér kernel,

$$F_L(x) = \sum_{|k| \le L}\left(1 - \frac{|k|}{L+1}\right) e^{ikx} = \frac{\sin^2\left((L+1)\frac{x}{2}\right)}{(L+1)\sin^2\frac{x}{2}}, \tag{A.1}$$

for real x.

We need the following result.

THEOREM A.1 [19, Theorem 8, Chapter 5]. *For $|k| \le N$, let*

$$c_k = \sum_{j=1}^M \rho_j z_j^k,$$

where $\rho_j \ge 0$ *and* $|z_j| = 1$. *Then, for all* L, $0 \le L \le N$,

$$(L+1)\,\|\rho\|_2^2 \le c_0^2 + 2\sum_{k=1}^L |c_k|^2.$$

Proof. Let $a_k = 1 - |k|/L+1$ be the coefficients of the Fejér kernel F_L and write $z_j = e^{i\theta_j}$. Since $\rho_j \ge 0$ and $F_L(\theta) \ge 0$ for all θ,

$$\sum_{|k| \le L} a_k |c_k|^2 = \sum_{|k| \le L} a_k \sum_{j,l} \rho_j \rho_l \left(\frac{z_j}{z_l}\right)^k$$

$$= \sum_{j,l} \rho_j \rho_l F_L(\theta_j - \theta_l) \ge F_L(0)\sum_{j=1}^M \rho_j^2 = (L+1)\sum_{j=1}^M \rho_j^2.$$

The theorem follows because $a_0 = 1$ and $a_k \le 1$. ∎

Proof of Theorem 2.2. We use (2.1) to extend the definition of c_k as $c_{-k} = \overline{c_k}$ for $k = 1, \ldots, N$ and $c_0 = \sum_{j=1}^{M} \rho_j$. We then define the Toeplitz matrix T_N, $(T_N)_{kj} = (c_{j-k})_{0 \le k,j \le N}$, and the polynomial

$$Q(z) = \prod_{j=1}^{M} (z - e^{i\gamma_j}) = \sum_{k=0}^{M} q_k z^k.$$

Then $q = (q_0, \ldots, q_M, 0, \ldots, 0)^t$

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