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EFFICIENT REPRESENTATION AND ACCURATE EVALUATION OF OSCILLATORY INTEGRALS AND FUNCTIONS

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Dedicated to Peter Lax

Abstract. We introduce a new method for functional representation of oscillatory integrals within any user-supplied accuracy. Our approach is based on robust methods for nonlinear approximation of functions via exponentials. The complexity of evaluation of the resulting representations of the oscillatory integrals does not depend or depends only mildly on the size of the parameter responsible for the oscillatory behavior.

1. Introduction.

Methods for asymptotic evaluation of oscillatory integrals have a long history ials.1585storylongtegralty parts in (1) yields the desired asymptotics. If at an isolated point $x^* \in [-1, 1]$ one or more derivatives of g vanish, then the asymptotics is obtained using the Taylor expansion of g at x^*. The specific powers in the asymptotic expansion depend e.g., on the type of stationary points of g. Asymptotics expansions of this type are also available in higher dimensions, see e.g., [34, 10].

More recently, Iserles and Norsett [26, 27] developed a Filon-type method for (1) by assuming that the amplitude function f is well approximated by polynomials.

2. Preliminaries. Our approach relies on algorithms for representing functions in [4] identifies the nodes of the generalized Gaussian quadratures in (3) as zeros of the *Discrete* Prolate Spheroidal Wave Functions (DPSWFs) [40], corresponding to small eigenvalues. The size of the eigenvalue determines the accuracy of the quadrature, While all $_j < 1$, $j = 0; 1; \dots$ $_j$

Since the coe cients r_{kl} in (14) are precomputed, this sequence of steps avoids the numerical di culties of nding the coe cients in (17

3. Representation of Fourier-type integrals. As an example of the straightforward use of our techniques, we consider a linear phase $g(x) = x$ in (1) and compute

$$I(\omega) = \int_{-1}^{1} f(x) e^{i\omega x} dx \tag{21}$$

assuming that f is well approximated by bandlimited exponentials with bandlimit c, where c ω. As in (17), for a target accuracy ϵ, we construct the approximation

$$f(x) \approx \sum_{m=1}^{M} c_m e^{ic\eta_m x}; \tag{22}$$

which immediately gives the explicit approximation

$$I(\omega) \approx \sum_{m=1}^{M} c_m e^{i(c\eta_m+\omega)} \operatorname{sinc}(c\eta_m + \omega): \tag{23}$$

Here the number of terms, M, is proportional to the bandlimit c and, therefore, the integral in (21) can be efficiently evaluated for any parameter ω at a cost independent of its size.

3.1. A representative example. This example illustrates our approach not only for a linear phase function

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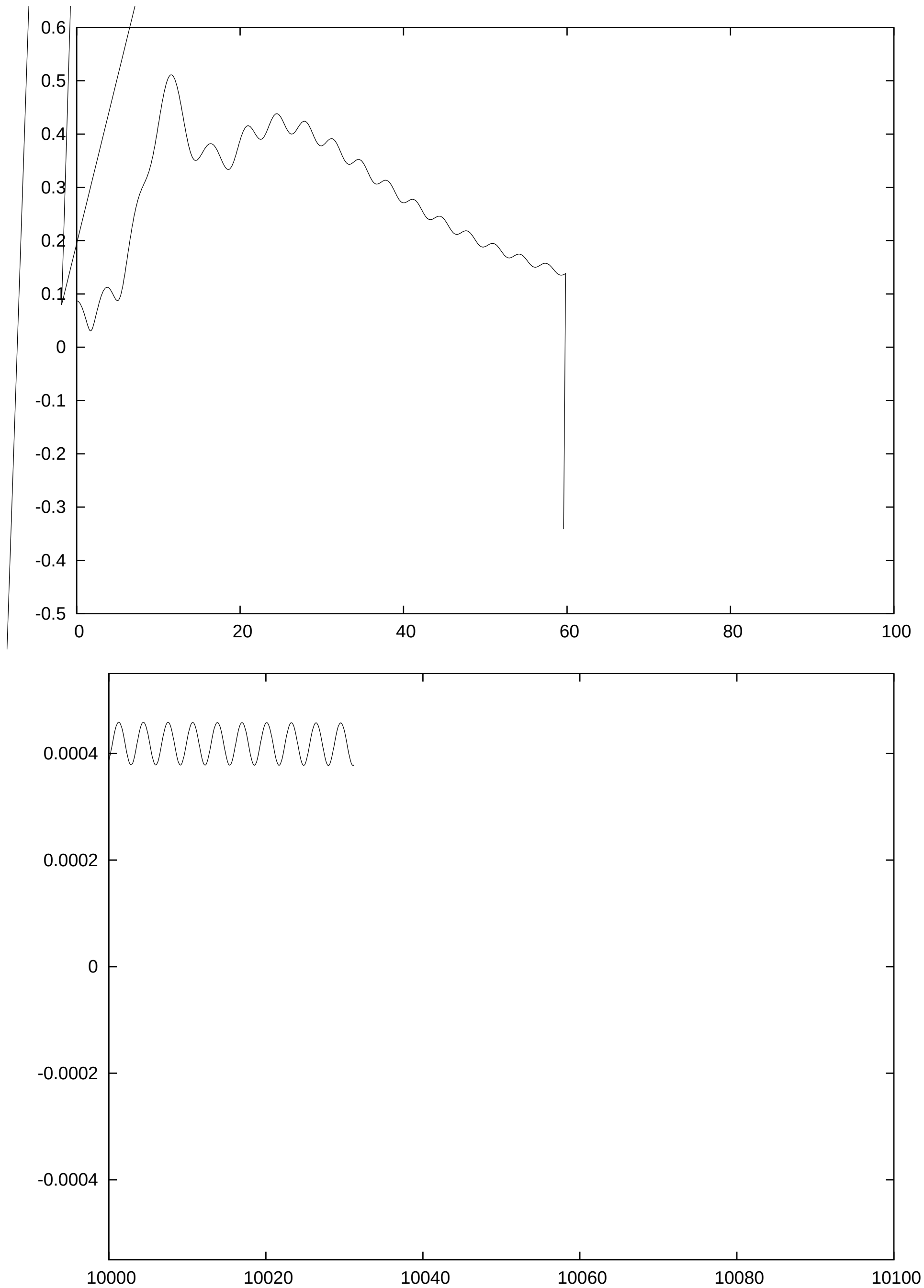

Figure 2. Approximation of the integral (21) for the amplitude f in (24) in the intervals $[0; 100]$ (top) and $[10000; 10100]$ (bottom). Real part of f is displayed with dashes, imaginary part with dots and absolute value with a solid line.

approach reduces the problem to the computation of the first few moments,

$$\int_a^b x^k e^{i\omega g(x)}dx,$$

which are assumed to be known. Also, to avoid computation of derivatives of f, a derivative-free variant is presented in [26].

We demonstrate how to construct a functional representation of such integrals on the canonical example

$$I^{(n)}(\omega) = \int_{-1}^{1} f(x)e^{i\omega x^n}dx, \quad \omega \ge 0, \tag{26}$$

where f is only mildly oscillatory and $n \ge 2$ is an integer. The only stationary point of this integral is at $x^* = 0$ and we subdivide the original interval to isolate the stationary point within a sufficiently small interval. We subdivide the interval as follows,

$$\begin{aligned}[-1,1] = {} & \left[-1,-2^{-1}\right] \cup \dots \cup \left[-2^{-l},-2^{-l-1}\right] \cup \dots \\ & \cup \left[-2^{-L-1},2^{-L-1}\right] \cup \dots \\ & \cup \left[2^{-l-1},2^{-l}\right] \cup \dots \cup \left[2^{-1},1\right]\end{aligned} \tag{27}$$

so that we approach the stationary point in a hierarchical fashion. The parameter L describing the number of levels of subdivision is chosen later. On all subintervals, except the one about zero, we perform a change of variables in order to use (25). We show below that, since the intervals become smaller when approaching the stationary point $x^* = 0$, the bandlimit of the integrand decreases exponentially fast. Once we reach a sufficiently small bandlimit, we evaluate the integral over $\left[-2^{-L-1},2^{-L-1}\right]$ directly. Hence, by first fixing the desired range of values of ω, the cost of evaluation depends only logarithmically on the maximum size of ω, i.e., it is proportional to the number of levels L in (27).

Since the intervals in (27) are symmetric about zero, we discuss only those where $x > 0$. Denoting

$$I_l^{(n)}(\omega) = \int_{2^{-l-1}}^{2^{-l}} f(x)e^{i\omega x^n}dx,$$

the change of variables $y = \left(2^{n(l+1)+1}x^n - (2^n+1)\right)/(2^n-1)$ yields

$$I_l^{(n)}(\omega) = \frac{e^{i\frac{(2^n+1)}{2^{n(l+1)+1}}\omega}}{2^{l+1}}\,\frac{1}{n}\left(\frac{2^n-1}{2}\right)^{\frac{1}{n}}\int_{-1}^{1} f\left(\frac{\left(\frac{2^n-1}{2}y+\frac{2^n+1}{2}\right)^{\frac{1}{n}}}{2^{l+1}}\right)\frac{e^{i\frac{(2^n-1)}{2^{n(l+1)+1}}\omega y}}{\left(y+\frac{2^n+1}{2^n-1}\right)^{\frac{n-1}{n}}}dy. \tag{28}$$

Hence, for any target accuracy, we can always find a value of L such that the contribution of $I_l^{(n)}(\omega)$ for $l > L$ is negligible. We note that the bandlimit of the exponential $e^{i\frac{(2^n-1)}{2^{n(l+1)+1}}\omega y}$ in (28) decreases exponentially fast as the parameter l

in0 G [(7 TJ/F10J/F11 6t2gZS52)]TJ/F467 2.6 Tf 3.39sr881(co3TJ/F18 5.9776 Tf831 9p33(band)]TF9 4.9813
−y]TJ-7.308.9620 Tf4.566 -8.07 Td [(2)]TJ/F10 5.9738 Tf8.855 3.616 Td [(n)]TJ/F14 959626 Tf7.637 -3.616 Td [(

and obtain

$$I_l^{(n)}(!) \quad \frac{e^{i \frac{(2^n+1)}{2^{n(l+1)+1}} !}}{2^l} \frac{1}{n} \quad \frac{2^n \quad 1}{2}^{\frac{1}{n}} \quad X$$

and substitute in (32) to obtain

$$
\begin{aligned}
F(\ ;p;a) &= \frac{1}{(\ )}\int_0^\infty t^{\ -1}e^{-at}\int_{-1}^{1} e^{(ip-t)y}\,dy\,dt \\
&= \frac{e^{ip}}{(\ )}\int_0^\infty \frac{t^{\ -1}e^{-(a+1)t}}{ip\quad t}\,dt \qquad \frac{e^{-ip}}{(\ )}\int_0^\infty \frac{t^{\ -1}e^{-(a-1)t}}{ip\quad t}\,dt:
\end{aligned}
$$

Using [35, 8.6.4], we arrive at

$$
F(\ ;p;a) = ie^{-i\left(ap+\frac{3}{2}\quad\right)}p^{1-}\quad[\quad(1\qquad;\quad i(a+1)p)\qquad(1\qquad;\quad i(a\quad 1)p)];
$$

where $(\ ;z$

where $-J \le j \le J$, $J = \lfloor (\frac{\omega}{\pi} - 1)/2 \rfloor$. Our assumption on ω implies that $J \ge 1$. We have

$$
\begin{aligned}
I_m^e(\omega) &= \sum_{j=-J}^{J} \int_{\frac{\pi}{\omega}(2j-1)}^{\frac{\pi}{\omega}(2j+1)} e^{ic_m x} e^{\beta \sin \omega(x+\alpha)} dx \\
&+ \int_{\frac{\pi}{\omega}(2J+1)}^{1} e^{ic_m x} e^{\beta \sin \omega(x+\alpha)} dx \\
&+ \int_{-1}^{-\frac{\pi}{\omega}(2J+1)} e^{ic_m x} e^{\beta \sin \omega(x+\alpha)} dx.
\end{aligned} \tag{37}
$$

We note that $J \sim \omega$ and that $\delta = (\pi/\omega)^{1/3}$. Changing variables

$$x = \frac{\pi}{\omega}(y + 2j),$$

in the integrals under the sum, we obtain

$$
\sum_{j=-J}^{J} \int_{\frac{\pi}{\omega}(2j-1)}^{\frac{\pi}{\omega}(2j+1)} e^{ic_m x} e^{\beta \sin \omega(x+\alpha)} dx =
\frac{\pi}{\omega} \left(\sum_{j=-J}^{J} e^{ic_m \frac{2\pi}{\omega} j} \right) \int_{-1}^{1} e^{ic_m \frac{\pi}{\omega} y} e^{\beta \sin(\pi y + \alpha\omega)} dy.
$$

For the integral over the interval $[\pi(2J+1)/\omega, 1]$, we change variables

$$x = \frac{\pi}{\omega}(py + q),$$

where

$$
\begin{aligned}
p &= \frac{\frac{\omega}{\pi} - (2J+1)}{2}, \\
q &= \frac{\frac{\omega}{\pi} + (2J+1)}{2}.
\end{aligned} \tag{38}
$$

We obtain

$$
\int_{\frac{\pi}{\omega}(2J+1)}^{1} e^{ic_m x} e^{\beta \sin \omega(x+\alpha)} dx = \frac{\pi p}{\omega} e^{ic_m \frac{\pi}{\omega} q} \int_{-1}^{1} e^{ic_m \frac{\pi}{\omega} py} e^{\beta \sin(\pi py + \pi q + \alpha\omega)} dy.
$$

Since

$$J \le \frac{\frac{\omega}{\pi} - 1}{2} < J + 1,$$

we observe that $p \in [0, 1)$ and, therefore, $p/(\omega/\pi) \in (0, \delta^3)$. For the integral over the interval $[-1, -\pi(2J+1)/\omega]$, the change of variables $x = -y$ reduces the problem to an integral of the previous type. Consequently, we arrive at

$$I_m^e(\omega) = \frac{\pi(2J+1)}{\omega} \sigma_m(\pi/\omega)\, u_m(\pi/\omega; \alpha\omega; \beta) + \frac{\pi p}{\omega} u_0(\pi/\omega; \alpha\omega; \beta);$$

where

$$
\sigma_m(\pi/\omega) = \frac{1}{2J+1} \sum_{j=-J}^{J} e^{ic_m \frac{2\pi}{\omega} j} = \frac{\sin\left(c_m \frac{\pi}{\omega}(2J+1)\right)}{(2J+1)\sin\left(c_m \frac{\pi}{\omega}\right)}; \tag{39}
$$

$$u_m(\pi/\omega; \alpha\omega; \beta) = \int_{-1}^{1} e^{ic_m \frac{\pi}{\omega} y} e^{\beta \sin(\pi y + \alpha\omega)} dy;$$

and

$$u_0(\ \omega,\ \omega,\ ) = e^{ic\ m\frac{q}{\omega}} \int_{-1}^{1} e^{ic\ m\frac{}{\omega}py} e^{\ \sin(\ py+\ q+\ \omega)} dy$$
$$+ \ e^{-ic\ m\frac{q}{\omega}} \int_{-1}^{1} e^{-ic\ m\frac{}{\omega}py} e^{-\ \sin(\ py+\ q-\ \omega)} dy.$$

The integrals $u_m(\ \omega,\ \omega,\ )$ are easy to evaluate using quadratures in [4] since, for large ω, the bandlimit of the integrand can be bound for large ω as shown below. Recall that, for small ω, the integral is evaluated directly. Unlike in [15], our approximation is not asymptotic and may be used for all $\omega \ \ 0$.

In order to estimate the bandlimit of the integrand in the representation of the functions u_m, it is enough to estimate the bandlimit of the function $h(y) = e^{\ \sin(qy)}$, for $q \in (0,\ ]$, $\ \in \mathbb{C}$, $\ \neq 0$, and $y \in [\ 1, 1]$. Using the expansion [1, 9.6.33] with $z =$ and $t = ie^{iqy}$, we obtain

$$e^{\ \sin qy} = \sum_{n\in\mathbb{Z}} i^{-n} I_n(\ ) e^{inqy},$$

where I_n is a modified Bessel function of order n, $I_n(\ ) = i^{-n} J_n(i\ )$. Therefore, for accuracy , it is enough to find $n_0 > 0$ such that

$$|I_n(\ )| < \ , n \ \ n_0,$$

yielding $q\ \ n_0$ as the estimate for the bandlimit. From the asymptotic expansion [1, 9.3.1] for large orders n, we obtain

$$I_n(\ ) \ \ \frac{1}{\sqrt{2\ n}} \left(\frac{e}{n}\right)^n \left(\frac{}{2}\right)^n .$$

Using Stirling's formula, we concludeF1 9.9626 Tf 6.428 17.89 17.8/F11 9. 0 Td.99.67121
n .

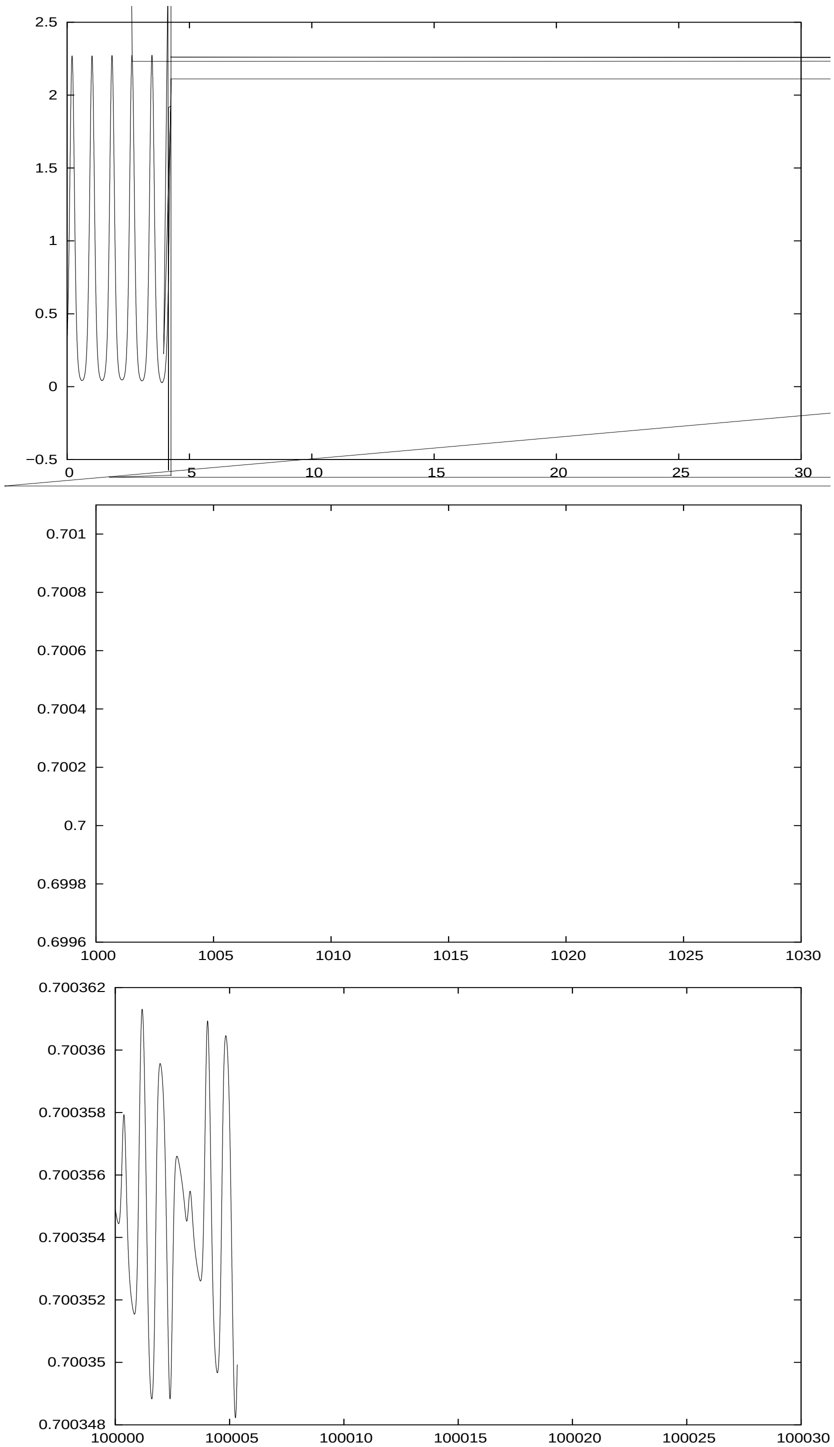

Figure 3. Evaluation of the integral (33), with parameters described in (4.1.1), for ! in the intervals [0; 30], [1000; 1030] and [100000;

4092 r4(G-18(R)1(EGOR)92(Y)-501(BEYL)1(KIN)-501(AND)-500(LUCAS)-500(MONZ)]TJ 254.913 1.763Td [(4)]TJ -1.02 6-1.763Td [(4OI

Next, substituting (42) into (4[illegible] the approximation

$$\hat{f}(\;;\;) \quad 2\;ab \sum_{m=1}^{M} w \,[\text{illegible}]\, \sqrt{\cdots a^2\cos^2 \;+ b^2\sin^2} \quad : \tag{43}$$

[illegible]d require a large number of terms [illegible]n. In contrast our nonlinear ap- [illegible] and only requires a small number of terms, M.

5.1. **One-dimensional oscillatory integral transforms.** We now consider the Fourier transform of a radial function $f(x) = f\left(\sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}\right)$ in dimension d. Since the Fourier transform of f is also radial,

$$\hat{f}(y) = u\left(\sqrt{y_1^2 + y_2^2 + \cdots + y_d^2}\right);$$

it is easy to see (e.g., by Bochn[illegible] [20, pp. 247]), that the univariate function $u(\;)$ is obtained via [illegible]sform,

$$[\text{illegible}]\; (t)^{-(\frac{d}{2}-1)} J_{\frac{d}{2}-1}(t)\, dt; \tag{44}$$

[illegible]tion of order $\frac{d}{2}$ 1 and 0. We note that if f has singularities, then the decay of u is slow. Writing u as

$$u(\;) = \int_0^{\infty} \tilde{f}(t)\,(t)^{-} J\,(t)\, dt; \tag{45}$$

where $= d{=}2 \quad 1$ and $\tilde{f}(t) = (2\;)^{\frac{d}{2}} f(t)t^{d-1}$, we observe that the kernel $(t)^{-}$ $J\,(t)$ is an oscillatory function. Instead of discretizing (45), we will approximate both, the function $\tilde{f}$ and the kernel by short sums of exponentials. As a consequence, we will obtain a rational representation for the function $u(\;)$.

First, by an analysis similar to the one in [2, p. 203], we express the kernel function $x^{-}\; J\,(x)$ as a Laplace type integral,

$$x^{-}\; J\,(x) = \int a(z)e^{-zx}dz = \int_{\mathbb{R}} a(\;(s))\;{}'(s)\,e^{-\;(s)x}ds \quad \sum_{m=1}^{M} a_m e^{-\;_m x}; \tag{46}$$

where the contour $= f\;\;(t) : t \in \mathbb{R}g$ is in the positive half plane, $a_m;\;_m \in \mathbb{C}$ with $\mathrm{Re}(\;_m) > 0$, $x > 0$, and

$$a(z) = \frac{2^{-}\;\left(1 + z^2\right)^{-1/2}}{p}$$

product of radial functions and spherical harmonics since this more general case can also be reduced to the evaluation of Hankel transforms [2, Thms 9.10.3 and 9.10.5].

Remark 6. We have several choices [4, 5, 6, 7] on how to e ciently approximate $\hat{f}$ in (50) and this decision depends on properties of the functions $\hat{f}$ and how we would like to represent the function u.

6. Conclusions.

As we have demonstrated, using nonlinear approximation of functions via exponentials (similarly, in other situations via Gaussians or rational functions) can drastically simplify the evaluation of oscillatory integrals. Indeed, as a result of such approximations, the integrals are evaluated explicitly and yield a functional representation within any user-selected accuracy.

Appendix.

Proof of Lemma 2.2.

Proof. We start by demonstrating that u in (7) can also be written as

$$u(x) = \sum_{l=1}^{M} e^{ic\ \ l}\ R_l(x): \tag{52}$$

Indeed, using (14) and that the matrix r_{kl} in (15) is symmetric, we obtain

$$\sum_{m=1}^{M} R_m(\ )e^{ic\ \ m x} = \sum_{m=1}^{M}\sum_{l=1}^{M} r_{ml}e^{ic\ \ l}\ e^{ic\ \ m x} = \sum_{l=1}^{M} e^{ic\ \ l}\left(\sum_{m=1}^{M} r_{lm}e^{ic\ \ m x}\right);$$

which yields (52).

Next, substituting $x = \ _m$ in (52), we obtain the exact collocation identity

$$e^{ic\ \ m} = u(\ _m);\ l = 1; : : : ; M: \tag{53}$$

De ning the function

$$(x) = \left| e^{ic\ \ x} - u(x)\right|^2 = 1 - \sum_{m=1}^{M} R_m(\ )e^{ic(\ _m - \ )x} - \sum_{m=1}^{M} \overline{R_m(\ )}e^{-ic(\ _m - \ )x}$$

$$+ \sum_{m;n=1}^{M} R_m(\ )\overline{R_n(\ )}e^{ic(\ _m - \ _n)x};$$

we observe that it is a linear combination of exponentials with bandlimit at most $2c$, so that we can write

$$(x) = \sum_{l=1}^{L} \ _l e^{2ic\ \ _l x};$$

with $j\ _l j \ 1$. Integrating (x) and approximating the integral by the quadrature (5), we derive the inequality

Z 1

*n*a9taTd [(a9taTd)]TJ9(y)-10 6.9730 6.973549(y)334(exact)-33resen9(y)quadrature

16 -4.114 9.3w3= 1

tJ/F7 6.9738 Tf 4.639 597(d [( )]TJ/F10 6.838 Tf 5.151 -1.494 Td [(l)]TJ)]

+L X = 1

=e6 . 47 d [(n)] T J 24x

and it remains to estimate the value of the constant $(\sum_l |\alpha_l|)^{\frac{1}{2}}$. Since

$$\sum_l |\alpha_l| \le 1 + 2\sum_{m=1}^{M} |R_m(\xi)| + \left(\sum_{m=1}^{M} |R_m(\xi)|\right)^2 = \left(1 + \sum_{m=1}^{M} |R_m(\xi)|\right)^2,$$

the results follows. □

Regarding an estimate of the L^∞ approximation error, the analysis in [4] assumed that the PSWFs have a uniform bound. However, the proven estimate (see [11, Theorem 3.1] and [36]) is

$$\|\psi_j\|_\infty \le \chi\sqrt{j+1}, \quad j \ge \frac{2c}{\pi}, \tag{54}$$

where $\chi \approx 2.35$. A possible improvement $\|\psi_j\|_\infty \le \sqrt{j+1/2}$ is suggested by the numerical evidence in [36, 37]. This potential growth of the uniform norm does not change the conclusion in [4] since the contribution of PSWFs with large indices is completely suppressed by the exponential decay of the corresponding eigenvalues. We have

Lemma 6.1. *For any target accuracy* $\epsilon > 0$ *and for any* $\xi \in [-1, 1]$ *consider the* and

$$\int_{-1}^{1} \sum_{l=1}^{M} w_l e^{ic\ _l(x-t)}\ _j(t)\,dt = \sum_{l=1}^{M} w_l e^{ic\ _l x}\overline{\ }_j\ _j(\ _l)$$

which, by (6), leads to the estimate

$$_j\ _j(x)\quad \frac{c\overline{\ }_j}{2} \sum_{l=1}^{M} w_l e^{ic\ _l x}\ _j(\ _l$$

d [( )]TJ/F11 9.9626 Tf 3.321 6-1.51[( ))]TJ/17 6.9738

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