DISTORTED-WAVE BORN AND DISTORTED-WAVE RYTOV APPROXIMATIONS

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The relation is considered between the distorted-wave Born (DWB) and the distorted-wave Rytov (DWR) approximations. Analyzing the Helmholtz equation, it is shown that the formal asymptotic justification of DWB and DWR approximations remains the same as that of the ordinary ones. A relation is derived between the first DWB and DWR approxima- [redacted]

This paper considers the relation between the dis- [redacted] (DWR) approximations. The ordinary Born [1] and [redacted] applications ranging from nuclear physics to seismic exploration (see refs. [4–7], for example). Within [redacted] known solution to a simpler equation. The only difference between the ordinary and distorted-wave ap- [redacted]

To illustrate this we consider the Helmholtz equation and show that the formal asymptotic justification of DWB and DWR approximations remains the same as that of the ordinary ones [3]. We also derive a relation between the first DWB and DWR approximations [redacted]

tion [redacted]

$$n^2(x) = n_0^2(x) + \epsilon n_1(x) + \epsilon^2 n_2(x) + \dots, \qquad (2)$$

where ϵ is a small parameter. The index of refraction [redacted] case of the ordinary Born and Rytov approximations [redacted] if we seek a solution of eq. (1) in the form [redacted] ficients of like powers of ϵ, we arrive at equations for the functions $U_j(x,k), j = 0,1 \dots$: [redacted]

$$(\nabla^2 + k^2 n_0^2)U_1 = -k^2 n_1 U_0,$$

$$(\nabla^2 + k^2 n_0^2)U_2 = -k^2 n_2 U_0 - k^2 n_1 U_1,$$

$$\dots \qquad (4)$$

[redacted] then have $U_0^{\pm}(x,k) = \exp(\pm ik x \cdot \nu)$, where ν is a unit [redacted]

We turn now to the Rytov approximation. The DWR

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approximation can be obtained if we seek a solution of eq. (1) in the form

$$U(x,k) = e^{ik\Phi(x,k)}, \tag{5}$$

where the phase function $\Phi(x,k)$ is a formal series

$$\Phi(x,k) = \Phi_0(x,k) + \epsilon\Phi_1(x,k) + \epsilon^2\Phi_2(x,k) + \dots \tag{6}$$

Using (5) and (1) we find that the phase function $\Phi(x,k)$ satisfies the equation

$$(\nabla\Phi)^2 - n^2 + (1/ik)\nabla^2\Phi = 0. \tag{7}$$

We now substitute the series (6) in (7), equate the coefficients of powers of ϵ, and arrive at equations for functions $\Phi_j(x,k), j = 0,1 \dots$:

$$(\nabla\Phi_0)^2 + (1/ik)\nabla^2\Phi_0 - n_0^2 = 0,$$

$$2\nabla\Phi_0 \cdot \nabla\Phi_1 + (1/ik)\nabla^2\Phi_1 - n_1 = 0,$$

$$2\nabla\Phi_0 \cdot \nabla\Phi_2 + (1/ik)\nabla^2\Phi_2 - n_2 + (\nabla\Phi_1)^2 = 0,$$

$$\dots \tag{8}$$

Eqs. (5) and (6) are the DWR approximation and eqs. (8) show how to compute the consecutive terms of the series for Φ. Let us now compare DWB and DWR approximations. It is easy to estimate the relative error of the mth DWR approximation. Indeed, it follows from (5) and (6) that

$$(U - U_R^m)/U = 1 - \exp\left(-ik \sum_{j=m+1}^{\infty} \epsilon^j \Phi_j\right)$$

$$= \mathcal{O}(ik\epsilon^{m+1}\Phi_{m+1}), \tag{9}$$

where U_R^m is the mth Rytov approximation,

$$U_R^m(x,k) = \exp\left(ik \sum_{j=0}^{m} \epsilon^j \Phi_j(x,k)\right).$$

To estimate the relative error of the DWB approximation we first establish relations between terms in series in (6) and (3). We have

$$U(x,k) = e^{ik\Phi_0} \sum_{d=0}^{\infty} \frac{1}{d!}\left(ik \sum_{j=1}^{\infty} \epsilon^j \Phi_j\right)^d$$

$$= e^{ik\Phi_0} \sum_{l=0}^{m} \epsilon^l \sum_{d=0}^{l} \frac{(ik)^d}{d!}$$

$$\times \sum_{j_1+j_2+\dots+j_d=1} \Phi_{j_1}\Phi_{j_2}\dots\Phi_{j_d}. \tag{10}$$

The mth DWB approximation is the sum of the $m + 1$ first terms in (10),

$$U_B^m = e^{ik\Phi_0} \sum_{l=0}^{m} \epsilon^l \sum_{d=0}^{l} \frac{(ik)^d}{d!} \sum_{j_1+j_2+\dots+j_d=l} \Phi_{j_1}\Phi_{j_2}\dots\Phi_{j_d}.$$

Thereby, we have

$$(U - U_B^m)/U = \mathcal{O}\left(\epsilon^{m+1} \sum_{d=0}^{m+1} \frac{(ik)^d}{d!}\right.$$

$$\left.\times \sum_{j_1+j_2+\dots+j_d=m+1} \Phi_{j_1}\Phi_{j_2}\dots\Phi_{j_d}\right) \tag{11}$$

Specifying the estimates (9) and (11) to the first DWB and DWR approximations, we have

$$(U - U_R^1)/U = 1 - \exp\left(-ik \sum_{j=2}^{\infty} \epsilon^j \Phi_j\right)$$

$$= \mathcal{O}(ik\epsilon^2\Phi_2), \tag{9a}$$

and

$$(U - U_B^1)/U = \mathcal{O}(\epsilon^2(ik\Phi_2 - \tfrac{1}{2}k^2\Phi_1^2)). \tag{11a}$$

When x and k are fixed, estimates in (9) and (11) demonstrate that both DWB and DWR approximations are of the same order of accuracy with respect to ϵ. Clearly, however, the errors in these two approximations will behave differently as functions of x and k.

Let us consider now the relation between the first DWB and the first DWR approximations. This relation for ordinary Born and Rytov approximations is of importance in linearized inverse scattering problems [7]. We set

$$\Phi_1 = e^{-ik\Phi_0} W_1 \tag{12}$$

and obtain from (8) that the function W_1 satisfies the

[redacted]

Also, from expressions (6) and (12) we have

$$\Phi_R^1 = \Phi_0 + \epsilon e^{-ik\Phi_0} W_1 . \quad (14)$$

Comparing (13) and the equation for the function U_1 in (4) and using (14) we arrive at the relation between the first DWB and DWR approximations,

[redacted]

mation,

$$U_B^1 = U_0 + \epsilon U_1 .$$

If $n_0(x) = 1$, relation (15) reduces to the well-known relation between classical Born and Rytov approximations [8].

The first DWR and DWB approximations are always [redacted] *timates in (9a) and (11a).* To show this, we provide a simple example. Since DWR and DWB do not differ from ordinary Rytov and Born approximations with respect to this property, our example deals with the [redacted]

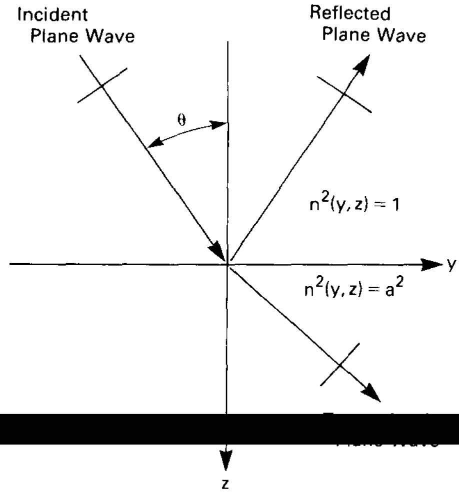

Fig. 1. Plane wave incident upon the interface between two homogeneous halfspaces.

[redacted]

$$n^2(y,z) = 1 + n_1(y,z), \quad (16)$$

where

$$n_1(y,z) \equiv n_1(z) = 0, \qquad z < 0;$$
$$= a^2 - 1, \quad z > 0 \quad (17)$$

and a is a positive constant. Comparing (16) with (2) [redacted]

$$\exp[ik(y \sin\theta + z \cos\theta)],$$

where θ is a fixed angle and k is the wave number, can be solved explicitly. We have the following expressions for the field

$$u(y,z) = \exp[ik(y \sin\theta + z\cos\theta)] + R \exp[ik(y \sin\theta - z \cos\theta)], \qquad z < 0;$$

[redacted] (18)

where

$$\alpha = (a^2 - 1)/\cos^2\theta .$$

[redacted]

$$R = \frac{1 - (1+\alpha)^{1/2}}{1 + (1+\alpha)^{1/2}}, \quad T = \frac{2}{1 + (1+\alpha)^{1/2}} .$$

To obtain the Rytov approximation to the field in (18) using a constant background with the index of refraction $n_0^2 = 1$ we first compute the phase of the background field. The phase of the background field is the phase of the plane wave which is as follows

$$\Phi_0 = y \sin\theta + z \cos\theta .$$

The first perturbation of the phase, the function Φ_1, depends only on z and satisfies the corresponding equation in (8) which in this case reduces to

[redacted]

where $n_1(z)$ is described in (17). $\Phi_1(z)$ and its normal derivative $\mathrm{d}\Phi_1(z)/\mathrm{d}z$ should be continuous at $z = 0$. Using these continuity conditions together with the

[redacted]

condition for the field to be outgoing for $z > 0$ we solve (19) and arrive at

$$\Phi_1(y,z) \equiv \Phi_1(z)$$

$$= -(\alpha/4ik)\exp(-2ikz\cos\theta), \qquad z<0;$$

[redacted]

Therefore, the first Rytov approximation to the field is a follows

[redacted]

$$u^{R}(y,z) = \exp[ik(y\sin\theta + z\cos\theta)$$

[redacted]

$$= \exp[ik(y\sin\theta + z\cos\theta) + \tfrac{1}{2}ikz\alpha\cos\theta - \tfrac{1}{4}\alpha], \quad z>0. \qquad (21)$$

Similar considerations of eq. (4) for the first Born approximation yield

$$u^{B}(y,z) = \exp[ik(y\sin\theta + z\cos\theta)] - \tfrac{1}{4}\alpha\exp[ik(y\sin\theta - z\cos\theta)], \quad z<0;$$

$$= [1-\tfrac{1}{4}\alpha(1-2ikz\cos\theta)] \times \exp[ik(y\sin\theta + z\cos\theta)], \qquad z>0. \qquad (22)$$

Eqs. (21) and (22) are obviously related through (15). However, for a given value of z, their accuracy is quite

[redacted]

matter how small the perturbation is. In contrast, the Rytov approximation (21) provides a reasonable answer.

The same conclusion about the behavior of Born and Rytov approximations can be drawn from estimates (9a) and (11a). Using corresponding equation in (8) we compute the function Φ_2 and obtain

[redacted]

$$= -\tfrac{1}{8}ikz\alpha^2\cos\theta + \tfrac{3}{32}\alpha^2, \qquad z>0.$$

[redacted]

tion (9a), the estimate of the relative error of the Born approximation (11a) has an extra term $\frac{1}{2}k^2\Phi_1^2$. It follows from (20) that this term is as follows

$$\tfrac{1}{2}k^2\Phi_1^2(z) = -\tfrac{1}{32}\alpha^2, \qquad z<0;$$

$$= \tfrac{1}{8}\alpha^2(kz\cos\theta - 1/2i)^2, \quad z>0,$$

which predicts much faster accumulation of error in the Born approximation compared to the Rytov approximation for the transmitted field ($z>0$).

References

[1] M. Born, Z. Physik 38 (1926) 803.

[redacted]

[8] A. Nayfeh, Perturbation methods (Wiley, New York, 1973).