The Inversion Problem and Applications of the Generalized Radon Transform

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Abstract

We prove that under certain conditions the inversion problem for the generalized Radon transform reduces to solving a Fredholm integral equation and we obtain the asymptotic expansion of the symbol of the integral operator in this equation.

We consider applications of the generalized Radon transform to partial differential equations with variable coefficients and provide a solution to the inversion problem for the attenuated and exponential Radon transforms.

Introduction

The classical Radon transform of a function u is the function $R_c u$ defined on a family of hyperplanes; the value of the $R_c u$ on a given hyperplane is the integral of u over that hyperplane.

and its applications were studied by I. M. Gel'fand [2], [3], F. John [8], S. Helgason [6], [7], D. Ludwig [12], P. D. Lax and R. S. Phillips [10], [11] and others (for more references see [7] for example). I. M. Gel'fand, M. I. Graev and N. Ya. Shapiro [3] introduced a general topological framework (the notion of double fibration) for generalized Radon transform R and its dual R^*. V. Guillemin [5] has shown that R^*R is an elliptic pseudodifferential operator. E. Quinto [14]

In Section 6 we define the transform R_λ which is a further generalization of [redacted]

In the following section we consider applications of the transform R_λ to [redacted]

We denote by $[s, \omega]$ elements of $\mathscr{H} = (\mathbb{R}^1 \times S^{n-1})/Z_2$. For given $x \in X$ let Y_x be the set of all hypersurfaces from the family $\mathscr{H}$ passing through the point $x \in X$,

$$Y_x = \{[\phi(x, \omega), \omega] \in \mathscr{H} \mid \omega \in S^{n-1}\}.$$

In order to have double fibration (see [3]), two conditions must be satisfied:

(v) if $H_{s,\omega} = H_{s',\omega'}$, then $[s, \omega] = [s', \omega']$.

(vi) if $Y_x = Y_{x'}$, then $x = x'$.

In our case, (v) and (vi) are consequences of the conditions (iii), (iv) and the implicit function theorem.

Now we can introduce the generalized Radon transform R and its dual R^*. [illegible]

$$(1.1) \qquad (Ru)([s, \omega]) = \int_{H_{s,\omega}} u(x) a(x, \omega) \Omega,$$

where Ω is the differential form

$$(1.2) \qquad \Omega = \sum_{j=1}^{n} (-1)^{j-1} \frac{\partial \phi(x, \omega)/\partial x^j}{|\nabla_x \phi(x, \omega)|^2} dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n.$$

The differential form Ω has been chosen to satisfy the following identity

$$(1.3) \qquad d_x \phi(x, \omega) \wedge \Omega = dx^1 \wedge \cdots \wedge dx^n.$$

The density $a(x, \omega)$ in (1.1) is a positive function on $X \times S^{n-1}$. We assume that the density $a(x, \omega)$ belongs to $C^\infty(X \times S^{n-1})$ and satisfies the condition $a(x, \omega) = a(x, -\omega)$.

2. The Fourier Integral Operator F

We denote by $a(x, \theta)$ and $b(y, \theta)$ the extensions of the densities $a(x, \omega)$ and $b(y, \omega)$ on the space $X \times (\mathbb{R}^n \setminus \{0\})$ by the formulae

$$a(x, \theta) = a(x, \omega),$$
$$b(y, \theta) = b(y, \omega),$$

where $\theta \neq 0$ and $\omega = \theta/|\theta|$.

Let $U(s)$ be an infinitely differentiable real function which has the following properties:

$$U(s) = U(-s)$$

and

$$|\partial_s^k U(s)| \leqq C(k)\langle s\rangle^{m-k},$$

where m is an arbitrary real number and $\langle s\rangle = (1+s^2)^{1/2}$.

We define the amplitude $A(x, y, \theta)$ by the formula

$$A(x, y, \theta) = a(x, \theta) b(y, \theta) U(|\theta|).$$

We can always choose the function U (by setting $U(|\theta|) = 0$ in the neighborhood of the point $\theta = 0$) in such a way that $A(x, y, \theta)$ belongs to $S^m(X \times X \times R^n)$, i.e., $A(x, y, \theta) \in C^\infty(X \times X \times R^n)$ and for every compact $Q \subset X \times X$ and for every three multiindices α, β, γ there is a constant $C_Q(\alpha, \beta, \gamma)$ such that

$$|\partial_\theta^\alpha \partial_x^\beta \partial_y^\gamma A(x, y, \theta)| \leqq C_Q(\alpha, \beta, \gamma)\langle\theta\rangle^{m-|\alpha|},$$

where $\langle\theta\rangle = (1+|\theta|^2)^{1/2}$.

For functions $u \in C_0^\infty(X)$ we introduce the Fourier integral operator F as follows:

$$(2.1) \qquad (Fu)(y) = \frac{1}{(2\pi)^n} \int_{R^n} \int_X e^{i\Phi(x,y,\theta)} A(x, y, \theta) u(x)\, dx\, d\theta,$$

where

(2.2) [illegible]

The standard procedure can be applied (see [16], [18]) to regularize the integral at the right-hand side of (2.1), if it is necessary.

Let us consider the set C_Φ,

$$C_\Phi = \{(x, y, \theta) \mid \nabla_\theta \Phi(x, y, \theta) = 0, x \in X, y \in X, \theta \in R^n \setminus \{0\}\}.$$

Remark. In further arguments we shall deal with amplitudes $A(x, y, \theta)$ which can write (3.1) as

$$F = R^* K R. \tag{3.3}$$

[redacted]

THEOREM 1. *The Fourier integral operator F in* (2.1) *can be factored into the form* (3.3), *where R is the generalized Radon transform* (1.1), *R^* its dual* (1.4), *and K is the operator with the kernel* (3.2)

[redacted]

Theorem 1 is a generalization of Theorem 1.1 of [12].

4. Inversion of the Generalized Radon Transform

In this section we prove that the operator F in (2.1) is a pseudo-differential operator. Further, we show that, given functions $\phi(x, \theta)$ and $a(x, \omega)$ in (1.1), we can define the density $b(y, \omega)$ in (1.4) and the function $U(r)$, so that the operator F will be "almost" the identity (up to a less singular operator).

Condition (iii) implies that F maps $C_0^\infty(X)$ continuously into $C^\infty(X)$. It also implies that the map defined by the integral in (2.1) can be extended as a continuous operator

$$F : \mathscr{E}'(X) \to \mathscr{D}'(X),$$

where $\mathscr{D}'(X)$ is the space of distributions on X (the dual of $C_0^\infty(X)$) and $\mathscr{E}'(X)$ is the space of distributions with compact support (the dual of $C^\infty(X)$). We shall say that an operator is regularizing if it maps $\mathscr{E}'(X)$ into $C^\infty(X)$.

Let $\mathscr{L}^m(X)$ be the class of standard pseudo-differential operators of order m. The Fourier integral operator belongs to $\mathscr{L}^m(X)$ if it has phase function

[redacted]

Proof: We can always find a function $\chi_\varepsilon(x, y) \in C^\infty(X \times X)$ such that $0 \leqq \chi_\varepsilon \leqq 1$ and

$$\chi_\varepsilon(x, y) = 1 \quad \text{if} \quad |x-y| < \tfrac{1}{2}\varepsilon,$$
$$\chi_\varepsilon(x, y) = 0 \quad \text{if} \quad |x-y| > \varepsilon,$$

for $\varepsilon > 0$. We can write the operator F as a sum $F = F_\varepsilon + \tilde{F}$, where

$$(F_\varepsilon u)(y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_X e^{i\Phi(x,y,\theta)} A(x, y, \theta) \chi_\varepsilon(x, y) u(x)\, dx\, d\theta,$$

and

$$(\tilde{F}u)(y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_X e^{i\Phi(x,y,\theta)} A(x, y, \theta)(1-\chi_\varepsilon(x, y)) u(x)\, dx\, d\theta.$$

We can now rewrite the operator F_ε:

$$(F_\varepsilon u)(y) = \frac{1}{(2\pi)^n} \iint \exp\{i(x-y)\cdot\xi + O(|x-y|^2|\theta|)\} A_0(x, y, \xi) u(x)\, dx\, d\xi,$$

where

$$A_0(x, y, \xi) = \frac{A(x, y, \theta(y, \xi))}{[illegible]}$$

[illegible]

Both conditions of Theorem 3 are satisfied, if ε is sufficiently small. Hence, $F_\varepsilon \in \mathscr{L}^m(X)$. Since $\tilde{F} \in \mathscr{L}^{-\infty}(X)$, we see that $F \in \mathscr{L}^m(X)$.

To prove the second part of Theorem 2 we need the following

LEMMA 1. *An operator $B \in \mathscr{L}^m(X)$, where $m < 0$, can be extended to a compact operator from $L^2(X$, compact$)$ to $L^2(X$, loc$)$.*

The proof of Lemma 1 can be found in [18].

To complete the proof of Theorem 2 it is sufficient to show that $F - I \in \mathscr{L}^m(X)$, for some $m < 0$. We shall do this in the next section.

Let us summarize our results. Theorems 1 and 2 reduce the inversion problem for the generalized Radon transform (1.1) to solving a Fredholm integral equation. More precisely, given the generalized Radon transform $v(s, \omega) = (Ru)(s, \omega)$, we can find the function u as a solution of the integral equation

[illegible]

$$u + Tu = R^* K v, \tag{4.1}$$

where the operator T is given in Theorem 2.

and

$$\left(F - \sum_{l=0}^{N} T_l\right) \in \mathscr{L}^{-(N+1)}(X) \quad \textit{for} \quad N = 0, 1, \cdots.$$

In particular,

$$(F - I) \in \mathscr{L}^{-1}(X).$$

Moreover, we have obtained the explicit formula (5.6) for the operator T_1.

COROLLARY 1. *If* $a(x, \omega) = h^{1/2}(x, \omega)$ *and, thereby,* $b(y, \omega) = h^{1/2}(y, \omega)$, *then* $(F - I) \in \mathscr{L}^{-2}(X)$.

Corollary 1 follows from (5.6).

6. The Generalized Radon Transform R_λ

The generalized Radon transform in (1.1) can formally be written as

$$(R(a)u)(s, \omega) = \int_{\mathbb{R}^n} u(x)a(x, \omega)\delta(s - \phi(x, \omega))\, dx, \tag{6.1}$$

where δ denotes the δ-function concentrated on the surface $s = \phi(x, \omega)$. We use the notation $R(a)$ to indicate explicitly the density $a(x, \omega)$.

We introduce the generalized Radon transform $R_\lambda(a)$ as follows:

$$(R_\lambda(a)u)(s, \omega) = \int u(x)a(x, \omega)\frac{(s - \phi(x, \omega))_+^\lambda}{\Gamma(\lambda+1)}\, dx, \tag{6.2}$$

where $(s - \phi(x, \omega))_+ = \max\{s - \phi(x, \omega), 0\}$, and Γ is the Gamma function.

where R_λ is the generalized Radon transform (6.2), *R^* is defined in* (1.4) *for $v \in C^\infty(R^1 \times S^{n-1})$, and K_λ is the convolutional operator with the generalized kernel*

$$K_\lambda(s) = \frac{e^{-i(\lambda+1)\pi/2}}{2(2\pi)^n} \int_{-\infty}^{+\infty} |r|^{n-1} U(r)(r+i0)^{\lambda+1} e^{irs}\, dr. \tag{6.3}$$

To prove Theorem 5 we repeat the proof of Theorem 1 taking into account that

$$e^{-i(\lambda+1)\pi/2}(\xi+i0)^{\lambda+1} \int_{-\infty}^{+\infty} e^{is\xi}(R_\lambda(a)u)(s,\omega)\, ds = \int_{\mathbb{R}^n} a(x,\omega)\, e^{i\xi\phi(x,\omega)}\, u(x)\, dx.$$

The generalized Radon transform defined in (6.2) has the following property:

$$\partial_s R_\lambda(a) = R_{\lambda-1}(a), \tag{6.4}$$

which we shall use in the next section, where we discuss applications of the transform R_λ to partial differential equations.

7. Applications of the Generalized Radon Transform to Partial Differential Equations

It is well known that the classical Radon transform R_c reduces the Cauchy problem for the wave equation with $n+1$ independent variables

$$\begin{aligned} &(\partial_t^2 - \Delta)u = 0, \\ &u(0,x) = f_1(x), \\ &u_t(0,x) = f_2(x), \end{aligned} \tag{7.1}$$

to the problem with two independent variables

$$\begin{aligned} &(\partial_t^2 - \partial_s^2)v = 0, \\ &v(0,s,\omega) = (R_c f_1)(s,\omega), \\ &v_t(0,s,\omega) = (R_c f_2)(s,\omega), \end{aligned} \tag{7.2}$$

[redacted]

$$Lu = \partial_{x_i}(\alpha_{ij}\partial_{x_j}u) + \beta_i\partial_{x_i}u + \gamma u,$$

where $\alpha_{ij} = \alpha_{ji}$, $\tilde{\alpha}|\xi|^2 \leqq \alpha_{ij}\xi^i\xi^j \leqq \tilde{\beta}|\xi|^2$, $0 < \tilde{\alpha} \leqq \tilde{\beta}$, and the operator formally adjoint to it,

$$L^*v = \partial_{x_j}(\alpha_{ij}\partial_{x_i}v) - \partial_{x_i}(\beta_i v) + \gamma v.$$

We denote

$$[u, w] = \alpha_{ij}\partial_{x_i}u\partial_{x_j}w,$$

and

$$\tilde{L}w = \partial_{x_j}(\alpha_{ij}\partial_{x_i}w) - \beta_i\partial_{x_i}w.$$

LEMMA 2. *The composition of the generalized Radon transform in* (6.2) *and the differential operator L can be written as*

$$R_\lambda(a)L = R_{\lambda-2}(a[\phi,\phi]) - R_{\lambda-1}(a\tilde{L}\phi + 2[a,\phi]) + R_\lambda(L^*a). \tag{7.3}$$

To prove Lemma 2 we observe that we can differentiate the function $(s-\phi(x,\omega))_+^\lambda/\Gamma(\lambda+1)$, i.e., the formal relation

$$\partial_{x_i}\frac{(s-\phi(x,\omega))_+^\lambda}{\Gamma(\lambda+1)} = -\frac{(s-\phi(x,\omega))_+^{\lambda-1}}{\Gamma(\lambda)}\partial_{x_i}\phi(x,\omega)$$

is well defined (see [4], for example). Thus, we can prove Lemma 2 integrating the expression $R_\lambda(a)Lu$ by parts.

We would like to make use of the relation in (7.3) and thus our first step is to simplify it. Let $\phi(x,\omega)$ in (6.2) be the solution of the eikonal equation

$$[\phi,\phi] = 1, \tag{7.4}$$

where

$$v(t, s, \omega) = (R_{\Lambda}^{N} u(t, x))(s, \omega),$$

for $N = 0, 1, 2, \cdots$.

The relation in (7.10) means that the second-order elliptic differential operator L can be represented by the operator ∂_s^2 (up to a smoothing operator) in a domain,

and

$$(R_\mu^{\text{odd}} u)(s, \omega) = \int_{\mathbb{R}^2} u(x) \sinh(\mu x \cdot \omega^\perp) \delta(s - x \cdot \omega)\, dx.$$

We introduce dual transforms $R_\mu^{*,\text{even}}$ and $R_\mu^{*,\text{odd}}$ as

$$(R_\mu^{*,\text{even}} v_+)(y) = \int_{|\omega|=1} \cosh(\mu y \cdot \omega^\perp) v_+(s, \omega)|_{s = y\cdot\omega}\, d\omega,$$

and

$$(R_\mu^{*,\text{odd}} v_-)(y) = \int_{|\omega|=1} \sinh(\mu y \cdot \omega^\perp) v_-(s, \omega)|_{s = y\cdot\omega}\, d\omega,$$

where the functions v_+ and v_- satisfy the relations $v_+(s, \omega) = v_+(-s, -\omega)$ and $v_-(s, \omega) = -v_-(-s, -\omega)$.

We consider the pseudo-differential operator F_μ,

(8.8) $$(F_\mu u)(y) = \frac{1}{(2\pi)^2} \iint \cosh\left(\mu(x-y)\cdot\frac{\theta^\perp}{|\theta|}\right) e^{i(x-y)\cdot\theta} u(x)\, dx\, d\theta,$$

where $\theta^\perp$ is the vector orthogonal to the vector θ: $\theta^\perp = (-\theta_2, \theta_1)$. The operator F_μ can be written as $F_\mu = F_\mu^+ - F_\mu^-$, where

$$(F_\mu^+ u)(y) = \frac{1}{(2\pi)^2} \iint \cosh\left(\mu x \cdot \frac{\theta^\perp}{|\theta|}\right) \cosh\left(\mu y \cdot \frac{\theta^\perp}{|\theta|}\right) e^{i(x-y)\cdot\theta} u(x)\, dx\, d\theta,$$

[redacted]

The proof of Lemma 3 can be found in the appendix.

Lemma 3 implies that the operator F_μ in (8.8) can be represented as $F_\mu = I + T_\mu$, where the operator T_μ has the kernel

$$\mu. \; I_1(\mu|x-y|).$$

[redacted]

Thereby, the inversion problem for the exponential Radon transform is reduced to solving the integral equation

$$(8.9) \qquad u(y) + \frac{\mu}{\;} \int \frac{I_1(\mu|x-y|)}{\;} u(x)\,dx = f(y)$$

[redacted]

We set $y=\cos\psi$; then $\sin\psi=(1-y^2)^{1/2}$ for $\psi\in[0,\pi]$, and $\sin\psi=-(1-y^2)^{1/2}$ for $\psi\in[\pi,2\pi]$. We obtain

[5] Guillemin, V., and Sternberg, S., *Geometrical Asymptotics*, Math. Surveys 14, Amer. Math. Soc., Providence, R.I, 1977.

[6] Helgason, S., *The Radon transform on Euclidean spaces, compact two-point homogeneous spaces*