# ON THE REPRESENTATION OF OPERATORS IN BASES OF COMPACTLY SUPPORTED WAVELETS\*

### G. BEYLKIN<sup>†</sup>

A str ct This paper describes exact and explicit representations of the di erential operators,  $n = 1 \ 2 \cdots$ , in orthonormal bases of compactly supported wavelets as well as the representations of the Hilbert transform and fractional derivatives. The method of computing these representations is directly applicable to multidimensional convolution operators.

Also, sparse representations of shift operators in orthonormal bases of compactly supported wavelets are discussed and a fast algorithm requiring (  $\log$  ) operations for computing the wavelet coe cients of all circulant shifts of a vector of the length =  $2^n$  is constructed. As an example of an application of this algorithm, it is shown that the storage requirements of the fast algorithm for applying the standard form of a pseudodi erential operator to a vector (see [G. Beylkin, R. R. Coifman, and V. Rokhlin, Comm. Pure. Appl. Math., 44 (1991), pp. 141{183]) may be reduced from ( ) to ( $\log^2$  ) signi cant entries.

 ${f ey}$  ords wavelets, di erential operators, Hilbert transform, fractional derivatives, pseudodi erential operators, shift operators, numerical algorithms

AMS\_MOS s ect c ssi c tions 65D99, 35S99, 65R10, 44A15

1. Introduction. n ID ec e n od ced co p c y ppo ed e e c p o ed o e e y ef n n e c n y n p pe e nd exact and explicit ep e en on of e e c ope o de e e n fo f e c n n o ono e of co p c y ppo ed e e e o p e en n O N o N o o n o fo co p n e e e coe c en of n n c c n f of ec o of e en n n n o n o p pe e on y co p e e n on n d d fo of ope o n ce p e e, o o, n n n d d fo, f o n e n o n o for the color of the c

econd e co p e e non nd d fo of e f ope o ope o po n n p c c pp c on of e e ec e e e e coe c en e no f n n n nce e non nd d nd nd d fo of ope o e p e nd e y o co p e no n e e e p e en on co pen e fo e c of f n nce, e e e e e p n on of f of ec o o of ce y e o ned y pp y n e f ope o d ec y o e coe c en of e o n e p n on e coe c en fo e f ope o y e o ed n d nce nd ed needed.

e o ono of co p c y ppo ed e e of  $\mathbf{L}^2$   $\mathbf{R}_{\mathbf{u}}$ ) fo ed y e d on nd n on of n e f nc on  $\mathbf{x}_{\mathbf{v}}$ )

$$j,k$$
  $\mathbf{X}_{i}$ )  $-j/2$   $-j$   $\mathbf{X}$   $\mathbf{K}_{i}$ );

ee $\mathbf{j}$ ;  $\mathbf{k}$   $\mathbf{2}$   $\mathbf{Z}$ . ef nc on  $\mathbf{x}$ , co p n on e c n f nc on  $\mathbf{x}$  nd eef nc on fy e fo o n e on

$$\mathbf{x}_{\mathbf{k}}$$
  $\mathbf{p}_{\mathbf{k}}$   $\mathbf{h}_{k}$   $\mathbf{x}$   $\mathbf{k}_{\mathbf{k}}$ ;

e e

$$\mathsf{g}_k$$
  $\mathsf{g}_{k-1}$ ;  $\mathsf{k}$  ; ;  $\mathsf{L}$   $\mathsf{d}$ ;

 $\operatorname{nd}$ 

$$Z_{+\infty}$$
  $X_{\nu}dx$   $A$ :

n dd on efnc on **M** n n o en

where

$$\mathsf{P} \; \mathsf{y}_{\mathsf{k}} \rangle \qquad \qquad \mathsf{P} \; \mathsf{y}_{\mathsf{k}} \rangle \qquad \qquad \mathsf{k} \qquad \qquad \mathsf{k} \qquad \qquad \mathsf{k} \qquad \mathsf{y}^{k}$$

and R is an odd polynomial such that

P 
$$\mathbf{y}_{k}$$
  $\mathbf{y}^{M}\mathbf{R}_{\frac{1}{2}}$   $\mathbf{y}_{k}$  fo  $\mathbf{y}$   $\mathbf{A}_{k}$ 

and

$$\operatorname{p}_{0 \leq y \leq 1} \mathsf{P} \mathsf{y}_{\hspace{-0.5mm}\boldsymbol{\mathsf{u}}}) \mathsf{y}^M \mathsf{R} \; \frac{1}{2} \; \mathsf{y}_{\hspace{-0.5mm}\boldsymbol{\mathsf{u}}}) \; < \; ^{2(M-1)} :$$

$$\mathsf{T} \quad \mathsf{fA}_j; \mathsf{B}_j;_{-j} \mathsf{g}_{j \in \mathbf{Z}}$$

c n on e p ce  $\mathbf{V}_j$  nd  $\mathbf{W}_j$ 

$$\mathbf{A}_{j} \quad \mathbf{W}_{j} \; \mathbf{I} \; \mathbf{W}_{j};$$

$$\mathsf{B}_{j} \; \mathsf{V}_{j} \; \mathsf{!} \; \mathsf{W}_{j};$$

$$\mathbf{w}_{j} \quad \mathbf{W}_{j} \mid \mathbf{V}_{j}$$
:

e ope o  $\mathbf{f}\mathbf{A}_j; \mathbf{B}_j; _{-j}\mathbf{g}_{j\in\mathbf{Z}}$  e de ned  $\mathbf{A}_j$   $\mathbf{Q}_j\mathsf{T}\mathbf{Q}_j$   $\mathbf{B}_j$   $\mathbf{Q}_j\mathsf{T}\mathbf{P}_j$  nd  $\mathbf{P}_j$   $\mathbf{P}_j\mathsf{T}\mathbf{Q}_j$  e e  $\mathbf{P}_j$  e p o ec on ope o on e p ce  $\mathbf{V}_j$  nd  $\mathbf{Q}_j$   $\mathbf{P}_{j-1}$   $\mathbf{P}_j$  e p o ec on ope o on e p ce  $\mathbf{W}_j$ .

e e e e en  $\frac{j}{il}$   $\frac{j}{il}$   $\frac{j}{il}$  of  $\mathbf{A}_j$   $\mathbf{B}_j$   $\mathbf{P}_j$  nd  $\mathbf{r}_{il}^j$  of  $\mathbf{T}_j$   $\mathbf{P}_j\mathsf{T}\mathbf{P}_j$   $\mathbf{i};\mathbf{l};\mathbf{j}$  2  $\mathbf{Z}$  fo e ope o  $\mathbf{d}$ = $\mathbf{d}\mathbf{x}$ 

### **OPERATORS IN BASES OF COMPACTLY SUPPORTED WAVELETS**

1.71

(1) If the integr





# **OPERATORS IN**

G. BEYLKIN

# **OPERATORS IN**

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A) If the integrals in A or A exist, then the coefficients  $\mathbf{r}_{l}^{(n)}$ ;  $\mathbf{l} \ \mathbf{2} \ \mathbf{Z}$  satisfy the following system of linear algebraic equations

 $\mathbf{r}_{l}^{(n)} \qquad {}^{n} \, 4\mathbf{r}_{2l} \qquad \mathbf{a}_{2k-1} \, \mathbf{r}_{2l-2k+1}^{(n)} \qquad \mathbf{r}_{2l+2k-1}^{(n)} ) 5 \, ;$ 

and

 $\mathbf{X}$   $\mathbf{I}^n \mathbf{r}_l^{(n)}$   $\mathbf{A}^{(n)} \mathbf{n}$ ;

where  $\mathbf{a}_{2k-1}$  are given in  $\mathbf{A}_{\mathbf{A}}$ .

Let M n A = ; where M is the number of vanishing moments in M if the integrals in M or M exist; then the equations M and M have a unique solution with a finite number of nonzero coefficients  $\mathbf{r}_{l}^{(n)}$ ; namely,  $\mathbf{r}_{l}^{(n)}$  6 for  $\mathbf{L}$   $\mathbf{I}$   $\mathbf{L}$  ; such that for even  $\mathbf{n}$ 

 $\mathbf{r}_{l}^{(n)}$   $\mathbf{r}_{-l}^{(n)}$ ;

 $imes \mathbf{I}^{2 ilde{n}} \, \mathbf{r}_l^{(n)} \quad ; \quad \mathbf{n} \quad \mathcal{A}; \quad ; \mathbf{n} = \mathcal{A};$ 

and

 $\overset{ imes}{\sim}$   $\mathsf{r}_{l}^{(n)}$ 

and for odd **n** 

 $\mathbf{r}_l^{(n)}$   $\mathbf{r}_{-l}^{(n)}$ 

 $\times$   $\mathbf{r}^{(n)}$  ;  $\mathbf{n}$   $\mathbf{n}$  ;  $\mathbf{n}$ 

$$a_1$$
 -;  $a_3$   $\frac{1}{-}$ ;

nd

$$\mathbf{r}_{-2}$$
  $\stackrel{\prime}{-}$ ;  $\mathbf{r}_{-1}$   $\stackrel{\prime}{-}$ 1;  $\mathbf{r}_{0}$  ;  $\mathbf{r}_{1}$   $\stackrel{\prime}{-}$ 1;  $\mathbf{r}_{2}$   $\stackrel{\prime}{-}$ 

e e of coe c en = ; A; ; A; A= v one of e nd d c o ce of , n e de ence coe c en fo e d de e e e no e on e e e e L e e e e o n n o en  $\mathbf{M}$  do no e e e e o de e ponen ee  $\mathbf{p}$  e ep e en on of e d de e e on y f e n e of n n o en M . Remark . Le de e n eq on ene z n o  $\mathcal{N}$  fo  $\mathbf{d}^n = \mathbf{d} \mathbf{x}^n$  d ec y fo -w e e e -w

$$\mathbf{r}_l^{(n)}$$
  $\mathbf{r}_l^{(n)}$   $\mathbf{j}^{\mathbf{j}}$   $\mathbf{k}^{\mathbf{j}}$   $\mathbf{k}^{\mathbf{j}}$   $\mathbf{k}^{\mathbf{j}}$   $\mathbf{k}^{\mathbf{j}}$   $\mathbf{k}^{\mathbf{j}}$ 

e efo e

$$\mathbf{x}_{\mathbf{k}}$$
  $\mathbf{x}_{\mathbf{k}}$   $\mathbf{x}_{\mathbf{k}}$   $\mathbf{x}_{\mathbf{k}}$   $\mathbf{x}_{\mathbf{k}}$   $\mathbf{x}_{\mathbf{k}}$   $\mathbf{x}_{\mathbf{k}}$   $\mathbf{x}_{\mathbf{k}}$   $\mathbf{x}_{\mathbf{k}}$ 

ее

$$\mathbf{r}_{\mathbf{v}}$$
  $\mathbf{r}_{l}^{(n)}\mathrm{e}^{\mathrm{i}l\xi}$ :

n e e on

 $\mathbf{m}_0 = \mathbf{v}^{\mathbf{m}} = \mathbf{v}$ 

n o e,  $\ell$  , nd de of  $\stackrel{\bullet}{\text{nl}}$  nd  $\stackrel{\bullet}{\text{nl}}$  ep e y o e e en nd odd nd ce

$$\mathbf{A}(\mathbf{v})$$
  $\mathbf{r}(\mathbf{v})$   $\mathbf{n}(\mathbf{j}\mathbf{m}_0 = \mathbf{v})\mathbf{j}^2\mathbf{r} = \mathbf{v}$   $\mathbf{j}\mathbf{m}_0 = \mathbf{v}$   $\mathbf{v})\mathbf{j}^2\mathbf{r} = \mathbf{v}$   $\mathbf{v}$ :

By con de n e ope o  $\mathbf{M}_0$  de n e od c f nc on

$$\mathbf{M}_{0}\mathbf{f}_{\mathbf{u}}) \qquad \mathbf{M}_{0}\mathbf{f}_{\mathbf{u}} \qquad \mathbf{j}\mathbf{m}_{0} = \mathbf{u}^{2}\mathbf{j}^{2}\mathbf{f} = \mathbf{u}^{2} \qquad \mathbf{j}\mathbf{m}_{0} = \mathbf{u}^{2} \qquad \mathbf{u}^{2}\mathbf{j}^{2}\mathbf{f} = \mathbf{u}^{2}\mathbf{j}^{2}\mathbf{j}^{2}\mathbf{f} = \mathbf{u}^{2}\mathbf{j}^{2}\mathbf{j}^{2}\mathbf{j}^{2}\mathbf{f} = \mathbf{u}^{2}\mathbf{j}^{2}\mathbf{j}^{2}\mathbf{j}^{2}\mathbf{f} = \mathbf{u}^{2}\mathbf{j}^{2}\mathbf{j}^{2}\mathbf{j}^{2}\mathbf{f} = \mathbf{u}^{2}\mathbf{j}^{2}\mathbf{$$

 $M_{\theta}$ r  $^{-n}$ r:

 $\mathbf{r}$  ne en eco of e ope o  $\mathbf{M}_0$  co e pond no e e en e  $^{-n}$  nd e efo e, and n/e ep e en on of e de e, n e e e eq en o and n/ono e c po yno o on of  $\mathbf{A}(\mathbf{c})$  nd ce e e ope o  $\mathbf{M}_0$  o n od ced n 7 nd e e e e p o e con de ed

Remark I e eo e c y e nde ood e de e ope o o o e ene y ope o o o eneo y o v e n e p c d on p econd one n e e e e n e c e dence n f c of n e e nce ep e en one of e d n l e of co p n n e e e e e f n ope o n p ce e c n p ce o n p ce fo c cy, en y e cond on n e e nde nd e o of e le nl e o e e n e above e e o d of cc cy. e nc de e on e e e ppe o y e p econd oned on y on p ce.e e no e epecond on n'dec ed ee dde e epo e of cond on n'de

on y o e nf o e o o ene y of e y o nd doe no ...éc cond on n' d e o o e c e .

# $\mathsf{D}_2^p - \mathsf{P} \mathsf{D}_2 \mathsf{P}$

e e  $\mathbf{P}_{il}$  il, j  $\mathbf{n}$   $\mathbf{n}$  nd e e  $\mathbf{j}$  c o en depend  $\mathbf{n}$  on  $\mathbf{i}$ ;  $\mathbf{l}$  o  $\mathbf{N}$   $\mathbf{N} = j^{-1}$   $\mathbf{n}$   $\mathbf{i}$ ;  $\mathbf{l}$   $\mathbf{N}$   $\mathbf{N} = j$   $\mathbf{n}$   $\mathbf{n}$   $\mathbf{P}_{NN}$   $\mathbf{n}$  e  $\mathbf{n}$  d  $\mathbf{co}$   $\mathbf{p}$  e  $\mathbf{e}$  o  $\mathbf{n}$  cond on  $\mathbf{n}$  e





# Table 3

Condition numbers of the matrix of periodized second derivative (with and without preconditioning

 $\mathbf{n}'$   $\mathbf{n}$ 

$$\mathbf{J}_{\mathbf{w}}$$
  $\mathbf{J}_{\mathbf{w}}$   $\mathbf{J}_{\mathbf{w}}$ 

o de o

$$\mathbf{1}_{\mathbf{A}\mathbf{A}}$$
  $\mathbf{1}_{\mathbf{Q}}$   $\mathbf{1}_{\mathbf{Q}}$ 

B fo  $(A_{N})$  fo o f o e e p c , ep e en on n  $(A_{N})$   $(A_{N})$ 

$$a_{2k-1}$$
 k  $a_{2m}$  fo  $a$  m M  $a$ :

nce e o en of efnc on n eq on low ed o one pon q d e fo fo co p n e ep e en on of con o on ope o on e ne c e fo o ned n e c y e e nne fo e pec c o ce of e e e de c ed n eqn low low e e e fed o en of efnc on n e efe o p pe fo e de lo e e e n od ce do e e o p o c fo co p n ep e en on of con o

e e p deno e p nc p e  $\mathbf{s}$   $\mathbf{x}$  e ep e en on of  $\mathbf{H}$  on  $\mathbf{V}_0$  de ned y e coe c en

$$r_t = \frac{Z_{\infty}}{-\infty} (x + L) H(y) x_t dx; \quad 12 Z;$$

c n n co peey de ne o e coe cen of e non nd d fo  $\mathbb{N}$  ey  $\mathbb{H}$   $\mathbf{f} \mathbf{A}_j; \mathbf{B}_j; \mathbf{g}_{j \in \mathbf{Z}} \mathbf{A}_j$   $\mathbf{A}_0$   $\mathbf{B}_j$   $\mathbf{B}_0$  nd  $\mathbf{g}_j$  e e e e en  $\mathbf{g}_{l-l}$  nd  $\mathbf{g}_{l-l}$  of  $\mathbf{A}_0$   $\mathbf{B}_0$  nd  $\mathbf{g}_0$  e co p ed f o e coe cen  $\mathbf{r}_l$ 

nd

 $_{i}$   $\swarrow^{1} \swarrow^{1} \swarrow^{1}$   $\mathsf{h}_{k} \, \mathsf{g}_{k} \, \mathsf{r}_{2i+k-k} :$ 

e coe c en  $\mathbf{r}_l$  **l 2 Z** n  $\mathbf{l}$  **a**  $\mathbf{r}_l$  fy e fo o n y e of ne e c eq on

 $\mathbf{r}_l \quad \mathbf{r}_{2l} \quad \stackrel{\mathbf{r}_{2l}}{=} \mathbf{a}_{2k-1} \mathbf{r}_{2l-2k+1} \quad \mathbf{r}_{2l+2k-1}$ 

e e e coe c en  $\mathbf{a}_{2k-1}$  e en n  $\mathbf{a}_{2k-1}$  e o n e y p o c of  $\mathbf{r}_l$  fo e l

 $r_l = \frac{\sqrt{1}}{l} = 0 \quad \frac{\sqrt{1}}{l^{2M}} :$ 12

By e n (x,y) n e of (x,y) (x,y)

e o n  $\mathbf{r}_l$   $\mathbf{r}_{-l}$  nd e  $\mathbf{r}_0$  , e no e e coe c en  $\mathbf{r}_0$  c nno e de e ned fo, eq on  $\mathbf{r}_l$  nd  $\mathbf{r}_l$  nd  $\mathbf{r}_l$  o nd  $\mathbf{r}_l$  e y poc cond on  $\mathbf{r}_l$  e coe c en  $\mathbf{r}_l$  e o p e e coe c en  $\mathbf{r}_l$  l 6 ny pec ed cc cy, e no e e e ene z on fo co p nd e coe c en of e z n fo n e d en on fo d.

Example e co p e ee e e e e e e e e n fo fo d.

Examp

#### Table 5

The coexcients  $\{_{\mathbf{p}^l}\}_l$ ,  $=-7\cdots$  14 of the fractional derivative =0.5 for Daubechies' wavelet with six vanishing moments.

|       |    | Coe cients      |    | Coe cients      |
|-------|----|-----------------|----|-----------------|
|       |    | <b>)</b> .l     |    | $\mathbf{r}^l$  |
| M = 6 | -7 | -2.82831017E-06 | 4  | -2.77955293E-02 |
|       | -6 | -1.68623867E-06 | 5  | -2.61324170E-02 |
|       | -5 | 4.45847796E-04  | 6  | -1.91718816E-02 |
|       | -4 | -4.34633415E-03 | 7  | -1.52272841E-02 |
|       | -3 | 2.28821728E-02  | 8  | -1.24667403E-02 |
|       | -2 | -8.49883759E-02 | 9  | -1.04479500E-02 |
|       | -1 | 0.27799963      | 10 | -8.92061945E-03 |
|       | 0  | 0.84681966      | 11 | -7.73225246E-03 |
|       | 1  | -0.69847577     | 12 | -6.78614593E-03 |
|       | 2  | 2.36400139E-02  | 13 | -6.01838599E-03 |
|       | 3  | -8.97463780E-02 | 14 | -5.38521459E-03 |
|       |    |                 |    |                 |

6. Shift operator on  $V_0$  and fast wavelet decomposition of all circulant shifts of a vector. Le  $\operatorname{con} \operatorname{de}$  f y one on e p ce  $V_0$  ep e en ed y e

$$\mathbf{t}_{i-j}^{(0)}$$
  $i-j,1;$ 
 $\mathbf{e}$   $\mathbf{e}$   $\mathbf{e}$   $\mathbf{e}$  onece  $\mathbf{y}$  o  $\mathbf{n}'$   $\mathbf{j}$   $\mathbf{k}$   $\mathbf{j}$   $\mathbf{e}$   $\mathbf{a}_n$  of  $\mathbf{j}$   $\mathbf{k}$   $\mathbf{j}$   $\mathbf{e}$   $\mathbf{k}$   $\mathbf{j}$   $\mathbf{k}$   $\mathbf{j}$   $\mathbf{k}$   $\mathbf{j}$   $\mathbf{k}$   $\mathbf{j}$   $\mathbf{k}$   $\mathbf{j}$   $\mathbf{k}$   $\mathbf{j}$   $\mathbf{k}$   $\mathbf{j}$   $\mathbf{j}$   $\mathbf{k}$   $\mathbf{j}$   $\mathbf{k}$   $\mathbf{j}$   $\mathbf{k}$   $\mathbf{j}$   $\mathbf{k}$   $\mathbf{j}$   $\mathbf{k}$   $\mathbf{j}$   $\mathbf{k}$   $\mathbf{k}$   $\mathbf{j}$   $\mathbf{k}$   $\mathbf{k}$   $\mathbf{j}$   $\mathbf{k}$   $\mathbf{j}$   $\mathbf{k}$   $\mathbf{j}$   $\mathbf{k}$   $\mathbf{k}$ 

e on y nonze o coe cen  $\mathbf{t}_l^{(j)}$  on e c cej e o e nd ce  $\mathbf{L}$   $\mathbf{I}$   $\mathbf{L}$   $\mathbf{A}$  o  $\mathbf{t}_l^{(j)}$ !  $\mathbf{l}$ ,  $\mathbf{0}$ ,  $\mathbf{j}$ !  $\mathbf{1}$ ,  $\mathbf{A}$  ne pe e fo o nd e con ne coe cen  $\mathbf{t}_l^{(j)}$   $\mathbf{j}$   $\mathbf{i}$ ,  $\mathbf{j}$ ;  $\mathbf{j}$  for e fope on  $\mathbf{D}$  ece e e e e e n nd o en e no e e f y n ne e o e none e ed y o e e f e o e e nonze o coe cen  $\mathbf{t}_l^{(j)}$   $\mathbf{l}$  o de e ne  $\mathbf{j}$   $\mathbf{l}$   $\mathbf{j}$   $\mathbf{l}$   $\mathbf{l}$   $\mathbf{j}$   $\mathbf{l}$   $\mathbf{j}$   $\mathbf{l}$   $\mathbf{j}$   $\mathbf{l}$   $\mathbf{l}$   $\mathbf{j}$   $\mathbf{l}$   $\mathbf{j}$   $\mathbf{l}$   $\mathbf{l}$   $\mathbf{j}$   $\mathbf{l}$   $\mathbf{j}$   $\mathbf{l}$   $\mathbf{l}$   $\mathbf{l}$   $\mathbf{j}$   $\mathbf{l}$   $\mathbf{$ 

Table 6 The coe cients  $\{{j \choose l}\}_{l=-L+2}^{l=L-2}$  for Daubechies' wavelet with three vanishing moments, where L= 6 and  $\vec{\vee}=$  1  $\cdots$  8.

|              |    | Coe cients           |              |    | Coe cients           |
|--------------|----|----------------------|--------------|----|----------------------|
|              |    | (j)<br>l             |              |    | ( <i>j</i> )         |
| √ = 1        | -4 | 0.                   | √ = <b>5</b> | -4 | -8.3516169979703E-06 |
|              | -3 | 0.                   |              | -3 | -4.0407157939626E-04 |
|              | -2 | 1.171875E-02         |              | -2 | 4.1333660119562E-03  |
|              | -1 | -9.765625E-02        |              | -1 | -2.1698923046642E-02 |
|              | 0  | 0.5859375            |              | 0  | 0.99752855458064     |
|              | 1  | 0.5859375            |              | 1  | 2.4860978555807E-02  |
|              | 2  | -9.765625E-02        |              | 2  | -4.9328931709169E-03 |
|              | 3  | 1.171875E-02         |              | 3  | 5.0836550508393E-04  |
|              | 4  | 0.                   |              | 4  | 1.2974760466022E-05  |
|              |    | •                    | ;            |    | 4 70504000404005 07  |
| √ = 2        | -4 | 0.                   | √ = 6        | -4 | -4.7352138210499E-06 |
|              | -3 | -1.1444091796875E-03 |              | -3 | -2.1482413927743E-04 |
|              | -2 | 1.6403198242188E-02  |              | -2 | 2.1652627381741E-03  |
|              | -1 | -1.0258483886719E-01 |              | -1 | -1.1239479930566E-02 |
|              | 0  | 0.87089538574219     |              | 0  | 0.99937113652686     |
|              | 1  | 0.26206970214844     |              | 1  | 1.2046257104714E-02  |
|              | 2  | -5.1498413085938E-02 |              | 2  | -2.3712690179423E-03 |
|              | 3  | 5.7220458984375E-03  |              | 3  | 2.4169452359502E-04  |
|              | 4  | 1.3732910156250E-04  |              | 4  | 5.9574082627023E-06  |
| <i>√</i> = 3 | -4 | -1.3411045074463E-05 | √ = <b>7</b> | -4 | -2.5174703821573E-06 |
|              | -3 | -1.0904073715210E-03 |              | -3 | -1.1073373558501E-04 |
|              | -2 | 1.2418627738953E-02  |              | -2 | 1.1081638044863E-03  |
|              | -1 | -6.9901347160339E-02 |              | -1 | -5.7198034904338E-03 |
|              | 0  | 0.96389651298523     |              | 0  | 0.99984123346637     |
|              | 1  | 0.11541545391083     |              | 1  | 5.9237906308573E-03  |
|              | 2  | -2.3304820060730E-02 |              | 2  | -1.1605296576369E-03 |
|              | 3  | 2.5123357772827E-03  |              | 3  | 1.1756409462604E-04  |
|              |    |                      |              |    |                      |

### 1-7-7

#### OPERATORS IN BASES OF COMPACTLY SUPPORTED WAVELETS

 $\sqrt{\phantom{0}} = 4 \quad -4 \quad -1.2778211385012E-05$  $\sqrt{\phantom{0}} = 8 \quad -4 \quad -1.2976609638869E-06$ -3 -7.1267131716013E-04 -3 -5.6215105787797E-05 -2 7.5265066698194E-03 -2 5.6059346249153E-04 -1 -4.0419702418149E-02 -1 -2.8852840759448E-03 0 0.99042607471347 0.99996009015421 1 5.2607019431889E-02 2.9366035254748E-03 2 -1.0551069863141E-02 2 -5.7380655655486E-04 3 1.1071795597672E-03 3 5.7938552839535E-05 4 2.9441434890032E-05 1.3777042338989E-06

ope o o o o e pc e n e co pe ed fo e coe cen  $\mathbf{t}_l^{(j)}$  fo e f ope o c n e o ed n d nce nd ed needed. c e o e e e e od of n' p ene of e f ope o depend on e pec c pp c on nd y e e fo d n ndc ed o e e e fo o n' n e p e of n pp c on e e n e d of co p n' f ope o e co p e po e f e dec e f lo fo e e e deco po on of c c n f of ec o nd en o o y e ed o ed ce o le eq e en of one of e o of en e e o e deco po on of e co po on of ec o of en e e e eq e  $\mathbf{O}$   $\mathbf{N}_l$  ope on nce e coe c en e no f n n e

fo o , e  $\mathbf{s}_k^{j-1}$   $\mathbf{k}$  ,  $\mathbf{a}$ ; ;  $^{n-j}$  e one of e eco of e e on epe o c e  $\mathbf{j}$  and co p e

$$\mathbf{s}_k^j$$
  $\mathbf{v}$   $\mathbf{h}_n \mathbf{s}_{n+2k-1}^{j-1};$ 

$$\mathbf{s}_k^j$$
  $\mathbf{h}_n\mathbf{s}_{n+2k}^{j-1};$ 

nd

$$\left(\mathbf{d}_{k}^{j}\right)$$
  $\mathbf{d}_{k}^{j}$   $\mathbf{d}_{n=0}^{j}$   $\mathbf{g}_{n}\mathbf{s}_{n+2k-1}^{j-1}$ 

$$\mathbf{d}_{k}^{j}$$
 ,  $\mathbf{d}_{k}^{j}$  )  $\mathbf{g}_{n}\mathbf{s}_{n+2k}^{j-1}$ :

$$\mathbf{v}_1 \qquad \mathbf{d}_k^1 \quad \mathbf{v}_2 ; \mathbf{d}_k^1 \quad \mathbf{v}_2$$

 $\operatorname{nd}$ 

се**ј** 🖊 ее

$$(u_1 \quad s_k^1 \quad v); s_k^1 \quad v_k^2;$$

e e  $\mathbf{d}_k^1$   $\mathbf{v}$   $\mathbf{d}_k^1$   $\mathbf{d}$   $\mathbf{s}_k^1$   $\mathbf{v}$  and  $\mathbf{s}_k^1$   $\mathbf{d}$  e co p ed for  $\mathbf{s}_k^0$  cco d n o [4]. On e econd c e j e e

$$\mathbf{v}_2 \qquad \mathbf{d}_k^2 \qquad \mathbf{v}$$
;  $\mathbf{d}_{\mathrm{e}} \ \ \mathrm{ge} \ \ \mathrm{8})$ 

ne ce fo o  $\mathbf{S}_k^1$   $\mathbf{v}$  nd  $\mathbf{S}_k^1$   $\mathbf{v}$  e no n po e coe cen fo odd nd e en f c e co ec n  $\mathbf{v}_2$  nd  $\mathbf{u}_2$  e c.

e e e co  $\mathbf{v}_1; \mathbf{v}_2; ; \mathbf{v}_n$  con n e coe cen e e coe cen e no o n zed eq en y no de o cce e e e ene e o e  $\mathbf{i}_{loc}$   $\mathbf{i}_s; \mathbf{j}_s$  nd  $\mathbf{i}_b$   $\mathbf{i}_s; \mathbf{j}_s$  n  $\mathbf{O}$   $\mathbf{N}_s$  o  $\mathbf{N}_s$  o

$$\mathbf{i}_{s} \qquad \mathbf{i}_{s} \overset{l=-1}{\underset{l=0}{\overset{l=-1}{\overset$$

nd

$$\mathbf{i}_b \; \mathbf{i}_s; \mathbf{j}_b 
angle \qquad \qquad \mathbf{i}_b \; \mathbf{i}_s; \mathbf{j}_b 
angle \qquad \qquad \mathbf{i}_{l=n-1} \; \mathbf{i}_s;$$

e e  $\mathbf{i}_b$   $\mathbf{i}_s$ ;  $\mathbf{j}_a$ ) f  $\mathbf{j}$   $\mathbf{n}$  en e  $\mathbf{i}_b$   $\mathbf{i}_s$ ;  $\mathbf{j}_a$ ) pon o e e nn of e eq. o of die ence  $\mathbf{n}$   $\mathbf{v}_j$ . Note yield ecology of  $\mathbf{v}_j$  and cere e ence  $\mathbf{i}_b$   $\mathbf{i}_s$ ;  $\mathbf{j}_a$ ) and  $\mathbf{i}_b$   $\mathbf{i}_s$ ;  $\mathbf{j}_a$ )  $\mathbf{n}_a$  ecology of e ed period ecology of eq. (e.e., and it is a sign of example of example of example of example of example of example of edges of example of edges of example of edges of

y con c n fo ny ed cc cw p e non nd do nd d fo nd e e y ed c n e co of pp y n o f nc on.

Le e e [ (1) ]

f e ope o T con o on en K x; x  $\mathbf{z}_{a}$  K  $\mathbf{z}_{a}$  f nc on of z on y e non nd d fo of con o on eq e o O of  $\mathbf{N}_{a}$  of o le ee e p, e o ec on e e e nd d fo of con o on A e n e y e nd d fo of K x; x  $\mathbf{z}_{a}$  K  $\mathbf{z}_{a}$  n e x nd z fo e con o on ope o con n no o e n O of  $\mathbf{N}_{a}$  n e n e e e ne depend on one e on y

### G. BEYLKIN

feno con cend do of  $\mathbf{K} \times \mathbf{x} \times \mathbf{z}$  neem of e on on nece y con o on where  $\mathbf{x}$  nd  $\mathbf{z}$  for period of e ope of nonece y con o on where  $\mathbf{x}$  nd  $\mathbf{z}$  for period of e ope of nonece y con o on where  $\mathbf{x}$  is a condition of  $\mathbf{x}$  on  $\mathbf{x}$  or  $\mathbf{$