On Multiresolution Methods
in Numerical Analysis

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ABSTRACT. As a way to emphasize several distinct features of the multiresolution methods based on wavelets, we describe connections between the multiresolution LU decomposition, multigrid and multiresolution reduction/homogenization for self-adjoint, strictly elliptic operators. We point out that the multiresolution LU decomposition resembles a direct multigrid method (without W-cycles) and that the algorithm scales properly in higher dimensions.

Also, the exponential of these operators is sparse where sparsity is defined as that for a finite but arbitrary precision. We describe time evolution schemes for advection-diffusion equations, in particular the Navier-Stokes equation, based on using sparse operator-valued coefficients. We point out a significant improvement in the stability of such schemes.

1 Introduction

Multiresolution methods have a fairly long history in numerical analysis, going back to the introduction of multigrid methods [10], [18] and even earlier [22]. A renewed interest in multiresolution methods was generated recently by the development of wavelet bases and other bases with controlled time-frequency localization [23], [20], [13], [19], [12], [2], [1], etc.. The introduction of these new tools allows us to relate numerical analysis with harmonic analysis and signal processing by the fundamental need of an efficient representation of operators and functions.

It is useful to compare the wavelet approach with the multigrid method (MG) and the Fast Multipole Method (FMM). For most problems the wavelet approach, FMM, and MG provide the same asymptotic complexity. The differences are typically in the "constants" of the complexity estimates. These differences will, most likely, diminish in the future.

A typical MG is a fast iterative solver based on a hierarchical subdivision. Hierarchical subdivision is also used in FMM which was initially proposed for computing potential interactions [21], [17]. This algorithm requires order N operations to compute all the sums

$$p_j = \sum_{i \neq j} \frac{q_i q_j}{|x_i - x_j|}, \quad \text{where} \quad x_i \in \mathbf{R}^3 \quad i, j = 1, \ldots, N, \tag{1}$$

and the n

We call (5)

3   Sparsity of Exponential Operators

If $\mathcal{L}$ is a self-adjoint, strictly elliptic operator then the operator $e^{\mathcal{L}t}$ is sparse in wavelet bases (for a finite but arbitrary precision) for all $t \geq 0$. This observation has a significant effect on the methods for solving PDEs.

Let us consider a class of advection-diffusion equations of the form

$$u_t = \mathcal{L}u + \mathcal{N}(u), \qquad x \in \Omega \subset \mathbf{R}^d, \tag{7}$$

where $u = u(x,t)$, $x \in \mathbf{R}^d$, $d = 1,2,3$ and $t \in [0,T]$ with the initial conditions,

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{8}$$

and the linear boundary conditions

$$\mathcal{B}u(x,t) = 0, \qquad x \in \partial\Omega, \quad t \in [0,T]. \tag{9}$$

In (7) $\mathcal{L}$ represents the linear and $\mathcal{N}(\cdot)$ the nonlinear terms of the equation, respectively.

Using the semigroup approach we rewrite the partial differential equation (7) as a nonlinear integral equation in time,

$$u(x,t) = e^{(t-t_0)\mathcal{L}}u_0(x) + \int_{t_0}^t e^{(t-\tau)\mathcal{L}}\mathcal{N}(u(x,\tau)) \, d\tau, \tag{10}$$

and describe a new class of time-evolution schemes based on its discretization. A distinctive feature of these new schemes is exact evaluation of the contribution of the linear part. Namely, if the nonlinear part is zero, then the scheme reduces to the evaluation of the exponential function of the operator (or matrix) $\mathcal{L}$ representing the linear part.

We note that the incompressible Navier-Stokes equations can

where $\widehat{f}$ denotes the Fourier transform of the function f. It is not difficult to show that the projection operator on the divergence free functions (the Leray projection) may be written with the help of the Riesz transforms,

$$\mathbf{P} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} - \begin{pmatrix} R_1^2 & R_1 R_2 & R_1 R_3 \\ R_2 R_1 & R_2^2 & R_2 R_3 \\ R_3 R_1 & R_3 R_2 & R_3^2 \end{pmatrix}. \tag{16}$$

Applying the divergence operator to (11), we obtain $-\Delta p = \sum_{k,\,=1}^{3} \partial_k \partial\, u_k u$ and an expression for pressure in terms of the Riesz transforms, $p = -\sum_{k,\,=1}^{3} R_k R\, (u_k u)$. Substituting the expression for the pressure into (11) and taking into consideration that the Riesz transforms commute with derivatives and, moreover, $R_k \partial\, = R\, \partial_k$, we obtain

$$\mathbf{u}_t = \nu \Delta \mathbf{u} - \mathbf{P}\Big(\sum_{m=1}^{3} u_m \partial_m \mathbf{u}\Big), \tag{17}$$

instead of (11) and (12). Equations (17) are now in the form (7), where $\mathcal{L} = \nu \Delta$ and $\mathcal{N}(\mathbf{u}) = -\mathbf{P}(\sum_{m=1}^{3} u_m \partial_m \mathbf{u})$. The transformation from (11) and (12) to (17) is well known and appears in a variety of forms in the literature. Here we followed a derivation presented by Yves Meyer at Summer School at Luminy in 1997.

The apparent problem with (17) for use in numerical computations is that the Riesz transforms are integral operators (which makes (17) into an integro-differential equation). Let us point out that the presence of the Riesz transforms does not create serious difficulties if we represent operators $R_j, j = 1, 2, 3$ in a wavelet basis with a sufficient number of vanishing moments (for a given accuracy). The reason is that these operators are nearly local on wavelets, and thus, have a sparse representation. This approximate locality follows directly from the vanishing moments property. Vanishing moments imply that the Fourier transform of the wavelet and its several first derivatives vanish at zero, and therefore, the discontinuity of the symbol of the Riesz transform at zero has almost no effect. The precise statements about such operators can be found in [6] and [5].

Finally, in rewriting (17) as $\mathbf{u}_t = \mathcal{L}\mathbf{u} + \mathcal{N}(\mathbf{u})$, we incorporate the boundary conditions into the operator $\mathcal{L}$. For example, $\mathbf{u} = \mathcal{L}^{-1}\mathbf{v}$ means that u solves $\mathcal{L}\mathbf{u} = \mathbf{v}$ with the boundary conditions $\mathcal{B}u = 0$. Similarly, $u(x,t) = e^{\mathcal{L}t}u_0(x)$ means that u solves $u_t = \mathcal{L}u$, $u(x,0) = u_0(x)$ and $\mathcal{B}u(x,t) = 0$.

Computing and applying the exponential or other functions of operators in the usual manner typically requires evaluating dense matrices and is highly inefficient unless there is a fast transform that diagonalizes the operator. For example, if $\mathcal{L}$ is a circulant matrix, then computing functions of operators can be accomplished using the FFT. It is clear that in this case the need of the FFT for diagonalization prevents one from extending this approach to the case of variable coefficients.

In the wavelet system of coordinates computing the exponential of self-adjoint, strictly elliptic operators always results in sparse matrices, and therefore, using the exponential of operators for numerical purposes is an efficient option [8].

Further development of the approach of [8] can be found in [9], where issues of stability of time-discretization schemes with exact treatment of the linear part

M	γ	β_0	β_1	β_2
1	Q_2	$Q_1 - Q_2$	0	0
2	$\frac{1}{2}Q_2 + Q_3$	$Q_1 - 2Q_3$		

for $\nu > 0$, together with an initial condition,

$$u(x,0) = u_0(x), \qquad 0 \le x \le 1, \qquad (25)$$

and periodic boundary conditions $u(0,t) = u(1,t)$. Burgers' equation is the simplest example of a nonlinear partial differential equation incorporating both linear diffusion and nonlinear advection. In [8] a spatially adaptive approach is used to compute solutions of Burgers' equation via

$$u_{n+1} = Q_0(\mathcal{L}\Delta t)u_n - \frac{\Delta t}{2}Q_1(\mathcal{L}\Delta t)\left[u_n \partial_x u_{n+1} + u_{n+1}\partial_x u_n\right]. \qquad (26)$$

We refer to [9] for the analysis of stability of ELP schemes.

4   Conclusions

The wavelet based algorithms described above are quite efficient in dimension one. Although algorithms described above scale properly with size in all dimensions, establishing ways of reducing operation counts remains an important task in dimensions two and three. This is an area of the ongoing research and the progress will be reported elsewhere.

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