Synchronization in large directed networks of coupled phase oscillators

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We , d he emergence of collec i e nchroni a ion in large direc ed ne ork of he erogeneo, o cilla or b generali ing he cla ical K, ramo o model of globall co, pled pha e o cilla or o more reali ic ne ork . We e end recen heore ical appro ima ion de cribing he ran i ion o nchroni a ion in large , ndirec ed ne ork of co, pled pha e o cilla or o he ca e of direc ed ne ork . We al o con ider he ca e of ne ork i h mi ed po i i e-nega i e co, pling reng h . We compare o, r heor i h n, merical im, la ion and nd good agreemen . – *2005 American Institute of Physics*. [DOI: 10.1063/1.2148388]

I. INTRODUCTION

The cla ical K, ramo o model[13,14] de cribe a collec ion of globall co, pled pha e o cilla or ha e hibi a ran i ion from incoherence o nchroni a ion a he co, pling reng h i increa ed pa a cri ical al, e. Since real orld ne ork picall ha e a more comple r, c , re han all- o-all co, pling,[15,16] i i na , ral o a k ha effec in erac ion r, c-, re ha on he nchroni a ion ran i ion. In Ref. 12, e , died he K, ramo o model allo ing general connec i i of he node , and fo, nd ha for a large cla of ne ork here i ill a ran i ion o global nchron a he co, pling reng h e ceed a cri ical al, e k_c. We fo, nd ha he cri ical co, pling reng h depend on he large eigen al, e of he adjacenc ma ri A de cribing he ne ork connec i i . We al o de eloped e eral appro ima ion de cribing he beha - ior of an order parame er mea , ring he coherence pa he ran i ion. Thi pa ork a re ric ed o he ca e in hich $A_{nm}=A_{mn}$. 0, ha i , , ndirec ed ne ork in hich he co, - pling end o red, ce he pha e difference of he o cilla or .

Mo ne ork con idered in applica ion are direc ed,[15,16] hich implie an a mme ric adjacenc ma ri , $A_{nm} \neq A_{mn}$. Al o, in ome ca e he co, pling be een o o cilla or migh dri e hem o be o, of pha e, hich can be repre en ed b allo ing he co, pling erm be een he e o - cilla or o be nega i e, A_{nm} 0. The effec ha he pre ence of direc ed and mi ed po i i e-nega i e connec ion can ha e on nchroni a ion i , herefore, of in ere . Here e ho ho o, r pre io, heor can be generali ed o acco, n for he e o fac or . We , d e ample in hich ei her he a mme r of he adjacenc ma ri or he effec of he nega- i e connec ion are par ic, larl e ere and compare o, r he- ore ical appro ima ion i h n, merical ol, ion .

Thi paper i organi ed a follo . In Sec. II e re ie he re , l of Ref. 12 for , ndirec ed ne ork i h po i i e

man he erogeneo, co, pled pha e o cilla or . Thi i , a ion
can be modeled b he eq, a ion

$$\dot{}_n = \ _n + k\sum_{m=1}^{N}$$

A eraging o er he freq, encie , one ob ain he *frequency distribution approximation* (FDA)

$$r_n = k\sum_m A_{nm} r_m \int_{\ 1}^{1} g(zkr_m)\sqrt{1\ \ z^2}\,dz. \tag{13}$$

The al, e of he cri ical co, pling reng h can be obained from he freq, enc di rib, ion appro ima ion b le ing $r_n \to 0^+$, prod, cing

$$r_n^{(0)} = \frac{k}{k_0}\sum_m A_{nm} r_m^{(0)}, \tag{14}$$

here $k_0 \equiv 2/[\ g(0)]$. The cri ical co, pling reng h h, corre pond o

$$k_c = \frac{k_0}{}, \tag{15}$$

here i he large eigen al, e of he adjacenc ma ri A and $r^{(0)}$ i propor ional o he corre ponding eigen ec or of A. B con idering per , rba ion from he cri ical al, e a $r_n = r_n^{(0)} + \ r_n$, e panding $g(zkr_m)$ in Eq. (13) o econd order for mall arg, men , m, l ipl ing Eq. (13) b $r_n^{(0)}$ and , ming o er n, e ob ained an e pre ion for he order parame er pa he ran i ion alid for ne ork i h rela i el homogeneo, degree di rib, ion [17]

$$r^2 = \left(\frac{1}{k_0^2}\right)\left(\frac{k}{k_c}\ \ 1\right)\left(\frac{k}{k_c}\right)^{\ 3}, \tag{16}$$

for 0 $(k/k_c)\ \ 1 \ll 1$, he2r4115.078hj/F6.768598406.2813Tm2764444111..9.9789j/F599Tc-307.9h37859.di rib, ion 6.913Tm276444

$$r = \sum_{n=1}^{N} r$$

he adjacenc ma ri i independen l cho en o be 1 i h probabili s and 0 i h probabili 1 s, and he diagonal elemen are e o ero.) E en ho, gh he ne ork con- r, c ed in hi a i direc ed, for mo node $d_n^{\text{in}} \approx d_n^{\text{o}}$. For $N=1500$ and $s=2/15$, Fig. 1(a) ho he a erage of he order parame er r^2 ob ained from n, merical ol, ion of Eq. (1) a eraged o er en reali a ion of he ne ork and freq, encie (riangle), he freq, enc di rib, ion appro ima- ion (FDA, olid line), and he mean eld heor (MFT, long da hed line) a a f, nc ion of k/k_c, here he re , l for he FDA and he MFT are a eraged o er he en ne ork reali- i a ion (no e, ho e er, ha he FDA and he MFT do no depend on he freq, enc reali a ion). (The per , rba ion heor Eq. (16) agreed i h he freq, enc di rib, ion appro ima ion and a lef o, for clari .) The error bar cor- re pond o one andard de ia ion of he ample of en reali- i a ion . We no e ha he larger error bar occ, r af er he ran i ion. When he al, e of he order parame er are a er- aged o er en reali a ion of he ne ork and he freq, encie , he re , l ho er good agreemen i h he freq, enc di rib, ion appro ima ion and he direc ed mean eld heor .

In order o , d ho ell o, r heor de cribe ingle reali a ion , e ho in Fig. 1(b) he order parame er r^2 ob ained from n, merical ol, ion of Eq. (1) for a par ic, lar reali a ion of he ne ork and freq, encie (bo e), he ime a eraged heor (hor da hed line), and he freq, enc di - rib, ion appro ima ion (olid line) a a f, nc ion of k/k_c. A can be ob er ed from he g, re, in con ra i h he ime a eraged heor , he freq, enc di rib, ion appro ima ion de ia e from he n, merical ol, ion (bo e) b a mall b, no iceable amo, n . Thi beha ior i ob er ed for he o her reali a ion a ell. We no e ha he FDA and MFT re , l are ir , all iden ical for all en reali a ion . On he o her hand, he TAT and he re , l from direc n, merical ol, ion of Eq. (1) ho dependence on he reali a ion. Since he FDA and MFT incorpora e he reali a ion of he connec- ion A_{nm}, b, no he freq, encie , e in erpre he ob er ed reali a ion dependence of he TAT and he direc ol, ion of Eq. (1) a indica ing ha he la er dependence i d, e prima- ril o , c , a ion in he reali a ion of he freq, encie ra her han o , c , a ion in he reali a ion of A_{nm}.

No e ha for o, r e ample $N=1500$ and $s=2/15$ implie ha on a erage e ha e $d^{\text{in}} \approx d^{\text{o}} \approx 200$. Th, for compari- on p, rpo e , e genera ed an , ndirec ed ne ork a fol- lo : S ar ing i h=a Eq.a9F54825ek-2 i h TDfrem reali a 4direc

, re , a in he , ndirec ed ca e, he al, e of he a erage of he order parame er ob ained from n, merical ol, ion of Eq. (1). The direc ed per , rba ion heor gi e a good appro i- ma ion for mall al, e of k clo e o k_c, a e pec ed. On he o her hand, he direc ed mean eld heor predic a ran i- ion poin hich i maller han he one ac , all ob er ed.

When n, mericall ol ing Eq. (32) b i era ion of Eq. (33), on ome occa ion a period o orbi a fo, nd in ead of he de ired ed poin . If e deno e he lef hand ide of Eq. (33) b z_n^{j+1} and he righ hand ide b $f(z_n^j)$, e fo, nd ha con ergence o a ed poin a facili a ed b replacing he righ hand ide b $[z_n^j + f(z_n^j)]/2$ and nding he ed poin of hi modi ed em.

In hi e ample, a lo co, pling reng h [ro, ghl $k/k_c \lesssim 4$, here k_c i comp, ed from Eq. (37)] he order parame er comp, ed from n, merical ol, ion of Eq. (1) i maller han ha ob ained from he TAT and FDA. A k increa e , ho e er, he TAT and FDA heorie cap , re he a mp o ic al, e of he order parame er r. We no e ha in hi ca e he a mp o ic al, e i larger han ha corre pond- ing o pha e locking [i.e., he one ob ained b e ing $_n=0$ in Eq. (35), $r \approx 0.54$ 0.46=0.08], hich e indica e b a hori on al do -da hed line in Fig. 4, and m, ch maller han $r=1$, he al, e corre ponding o no fr, ra ion [i.e., $_n$ $_m$ $=0$ for A_{nm} 0 and for A_{nm} 0 in Eq. (35)]. The mall cale of he hori on al a i i d, e o he fac ha e are plo ing r^2, and o o, r de ni ion of he order parame er hich a ign a al, e of 1 o a nonfr, ra ed con g, ra ion. The mall al, e of he order parame er indica e a rong fr, ra ion.

We no e ha in hi e ample, in con ra i h he e - ample di c, ed o far, here i aria ion in he al, e of he order parame er predic ed b he FDA for differen reali a- ion of he ne ork. Thi indica e ha , a he e pec ed al, e of he co, pling reng h A_{nm} become mall (i.e., $|q \quad 1/2|$ mall), , c , a ion d, e o he reali a ion of he ne - ork become no iceable. Al ho, gh he al, e predic ed b he FDA and TAT depend on he reali a ion of he ne ork and freq, encie , e no e for k/k_c 6 ha he e al, e rack he al, e ob er ed for he n, merical im, la ion of he cor- re ponding reali a ion. A an ill, ra ion of hi , e plo in Fig. 5 he al, e of r^2 ob ained from he TAT (ar) and he al, e of r^2 ob ained from he FDA (diamond) er , he al, e ob ained from n, merical ol, ion of Eq. (1) for k/k_c =8. Each poin corre pond o a gi en reali a ion of he ne - ork, i h re , l a eraged o er en reali a ion of he fre- q, encie . The ellip e , rro, nding he ar (TAT da a) ha e er ical and hori on al half- id h corre ponding o he an- dard de ia ion of r^2 (TAT) and r^2 (im, la ion) for he en freq, enc reali a ion . The half- id h of he hori on al bar on he diamond (FDA da a) indica e he andard de ia ion of r^2 (im, la ion)

la ion in ne ork i h a m, ch larger n, mber of connec- ion per node, a he effec of , c , a ion o, ld likel be red, ced.

We al o con idered a ca e in hich he adjacenc ma ri i a mme ric and ha mi ed po i i e-nega i e connec ion . For $N=1500$ node , e con r, c ed an adjacenc ma ri b e ing i nondiagonal en rie o 1, 1, and 0 i h probabil- i 8/45, 4, 45, and 11/15, re pec i el . The la er probabil- i ield an e pec ed n, mber of connec ion of 400. O, r heorie ork a i fac oril in hi ca e, and, ince he re , l are imilar o ho e in Fig. 3, e do no ho hem. In hi ca e here i no g, aran ee ha here i a real eigen al, e [a needed for e ima ing he cri ical co, pling reng h in Eq. (15)], or ha he large real eigen al, e (if here i one) ha he large real par . N, mericall , e nd ha for ma rice con r, c ed a in hi e ample here i a real po i i e eigen- al, e and ha , f, r hermore, i i ell epara ed from he large real par of he remaining eigen al, e (ee Fig. 6). We al o nd hi for o her al, e of q pro ided $|q \ \frac{1}{2}|$ i no oo mall. We pro ide a di c, ion of hi i , e and ho he

of he non ero en rie being cho en randoml (e.g., in he mme ric ca e, he po i ion of he non ero en rie i cho en hen con r, c ing he ne ork , ing he con g, ra ion model), and heir al, e being al o de ermined randoml from a gi en probabili di rib, ion (e.g., 1 i h probabili q and 1 i h probabili 1 q). O, r in ere i foc, ed on he gap be een he large real eigen al, e (if here i one) and he large real par of he o her eigen al, e . In Ref. 23 he pec r, m of cer ain large par e ma rice i h a erage eigen al, e 0 and ro , m $\Sigma_{m=1} A_{nm}=1$ a de cribed and a he, ri ic anal ical approach a propo ed. U ing re , l for ma rice i h ero mean Ga, ian random en rie ,[24] Ref. 23 predic ha he pec r, m of he non-Ga, ian random ma- rice he con ider con i of a ri ial eigen al, e $=1$ i h he remaining eigen al, e di rib, ed , niforml in a circle cen ered a he origin of he comple plane i h radi,

$$= \sqrt{N} \quad , \tag{A1}$$

here ^{2} i he ariance of he en rie of he ma ri . We nd ha hi approach al o , cceed in de cribing he pec r, m of he ma rice in o, r e ample . In o, r ca e, he diagonal en rie are 0, o ha he a erage eigen al, e i al o 0 a in Ref. 23. We nd ha here i al a a large real eigen al, e appro ima el gi en b he mean eld al, e

$$= \langle \tilde{d}^2 \rangle / \langle \tilde{d} \rangle \tag{A2}$$

(ee Ref . 12 and 25), here $\tilde{d}_n = \Sigma_{m=1}^{N} A_{nm}$ and $\langle \tilde{d}^2 \rangle = \Sigma_{n=1}^{N} \tilde{d}_n^2$, hich in he ca e con idered in Ref. 23 red, ce o $=1$. We al o n, mericall con rm ha he remaining eigen al, e are , niforml di rib, ed in a circle of radi, a de cribed in Ref. 23. Thi i ill, ra ed in Fig. 6.

Th, for N if here i a gap of i e be- een he large real eigen al, e and real par of he re of he eigen al, e pec r, m. U ing Eq . (A1) and (A2) i can be ho n ha , for ne ork i h large eno, gh n, mber of con- nec ion per node or i h eno, gh po i i e (or nega i e) bia in he co, pling reng h, here i a ide epara ion be een he large eigen al, e and he large real par of he remain- ing eigen ec or . For mme ric ma rice , imilar re , l ap- pl (i.e., he b, lk of he pec r, m of he ma ri A can be appro ima el ob ained a de cribed abo e , ing Wigner' emicircle la).

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