RESEARCH ARTICLE | MARCH 09 2023

Sabina Adhikari ; Juan G. Restrepo ; Per Sebastian Skardal

33, 033116 (2023)
https://doi.org/10.1063/5.0116747

to obtain low-dimensional descriptions of the system's order parameters in terms of the hypergraph's structural properties. We illustrate our approach with two examples of a hypergraph with interactions of sizes 2 (links) and 3 (triangles): a random hypergraph and a hypergraph constructed in such a way that the numbers of links and triangles at each node are correlated. We derive analytical conditions on the properties of these hypergraphs that result in synchronization, incoherence, or bistable behavior and validate our results with numerical simulations.

This paper is organized as follows. In Sec. II, we present our hypergraph generative model and the Kuramoto model on hypergraphs. In Sec. III, we use the Ott–Antonsen ansatz and a mean-field approximation to obtain low-dimensional descriptions for the local and global order parameters. In Sec. IV, we demonstrate our framework on two example hypergraphs. In Sec. V, we discuss our results and their limitations.

II. MODEL

In this section, we introduce the hypergraph generative model and the Kuramoto model on hypergraphs.

A. Hypergraph model

A hypergraph is a pair of nodes and hyperedges (V, E), where V is the set of nodes labeled $n = 1, 2 \ldots, N$, and the set of hyperedges E is a set of subsets of V. The th order degree of a node n is given by $k_n^{( )}$, which gives the number of hyperedges with size that node n is a part of. The *hyperdegree* of node n is given by $\mathbf{k}_n = \{k_n^{(1)}, k_n^{(2)}, \ldots, k_n^{(M)}\}$, where M is the largest hyperedge size. For simplicity, we refer to hyperedges of sizes 2 and 3 as *links* and *triangles*, respectively. We denote by $N(\mathbf{k})$ the number of nodes with hyperdegree $\mathbf{k}$, and define the hyperdegree distribution as $P(\mathbf{k}) = N(\mathbf{k})/N$.

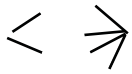
We will consider synchronization on a class of hypergraphs produced by the following generative model. For a given set of nodes $n = 1, 2, \ldots, N$

in Ref. 12, so here we focus for simplicity on the form of the interactions in Eq. (3).

III. DIMENSIONALITY REDUCTION

In this section, we use the Ott–Antonsen ansatz[28] to derive a low-dimensional description of the dynamics and use it to find

$$R^{(2)}(\mathbf{k}) = \sum_{\mathbf{k}',\mathbf{k}''} N(\mathbf{k}')N(\mathbf{k}'')a^{(3)}(\mathbf{k},\mathbf{k}',\mathbf{k}'') \int g(\omega')$$



absence of hyperedges of size larger than 2, we recover a smooth transition between incoherent and synchronized states as found in the standard network Kuramoto model.[25,43] Sufficiently strong higher-order interactions lead to an abrupt transition and bistability of incoherent and synchronized states (see Ref. 45 for a broader perspective of this issue). For a hypergraph with correlated links and triangles, we showed that the onset of synchronization and the onset of bistability depend on the moments of the degree distribution. For the hypergraph model we considered, higher-order interactions only affect the onset of bistability, but not the onset of synchronization (however, see additional discussion on this point below). We have also verified that similar results hold true for networks with power law and bimodal degree distributions.

The main limitations of our study are the requirement for hypergraphs to be produced by the generative model of Sec. II, the use of the mean-field approximation, and the use of approximation (9). [Here, we refer to the approximation that all nodes with the same hyperdegree are statistically equivalent as the mean-field approximation, rather than neglecting pair correlations in Eq. (9)]. The generative model we used assumes that the presence of a hyperedge connecting a group of nodes depends only on a set of pre-determined quantities of these nodes, which might not capture the generative mechanisms behind some real-world or model hypergraphs. For example, a simplicial complex model where triangles only join triples of nodes that are already forming a clique (as assumed in some studies[10,46]) is not included in the class of models that the generative model in Sec. II covers. In such a model, correlations between the states of nodes belonging to the same triangle could be non-negligible, and, thus, approximation (9) could break down. In that case, the techniques introduced in Ref. 42 could be needed to account for pair correlations. For example, for the SIS model on a simplicial complex, Ref. 47 finds that the epidemic threshold is only predicted correctly when accounting for pair correlations. In addition, Ref. 48 recently noted that synchronization properties in the strongly synchronized regime differ between simplicial complexes and random hypergraphs. Exploring the limitations and possible extensions of our method for simplicial complexes is an interesting problem left for future work.

Despite the limitations discussed above, our framework constitutes a flexible method to study the synchronization of phase oscillators on complex hypergraphs. While we demonstrated our framework in a particular case [hypergraphs constructed following Eqs. (27) and (28)], we emphasize that the techniques presented here allow for the study of a much larger class of systems. Examples include hypergraphs with independently chosen link and triangle degree distributions, correlations between hyperedge degrees and frequencies, and varying degrees of correlations between link and triangle degrees. The techniques presented here open a way to

[13] A. P. Millán, J. J. Torres, and G. Bianconi, "Explosive higher-order Kuramoto dynamics on simplicial complexes," Phys. Rev. Lett. 124