ßuid conduits have been observed experimentally [2,4], but their properties have never been studied.

scale $L$ is proportional to the uniform conduit radius while vertical variations are assumed to be weak according to

$$\tilde{r} = r/L, \quad \tilde{z} = {}^{1/2} z/L, \quad L = R_0/\sqrt{8}. \qquad (18)$$

The proportionality constant in the characteristic length $L$ is chosen for convenience in working with the governing equations but will be rescaled to arrive at the standard form of the conduit equation (1). The boundary is now denoted by $r = (z,t) = R_0 + R(z,t)$ or $\tilde{r} = (R_0 + R(z,t))/L \quad \tilde{R}(\tilde{z},\tilde{t})$. Hence the unit normal and tangent vectors for the conduit are given by

$$\hat{n}_c = \frac{1}{\tilde{n}_c} \begin{pmatrix} -1 \\ {}^{1/2}\frac{\tilde{R}}{\tilde{z}} \end{pmatrix}, \quad \hat{t}_c = \frac{1}{\tilde{t}_c} \begin{pmatrix} {}^{1/2}\frac{\tilde{R}}{\tilde{z}} \\ 1 \end{pmatrix}, \qquad (19a)$$

where

$$\tilde{n}_c = \tilde{t}_c = \left[1 + \left(\frac{\tilde{R}}{\tilde{z}}\right)^2\right]^{1/2}. \qquad (20)$$

Velocities are normalized to the radially averaged vertical velocity of the uniform conduit

$$\tilde{u}^{(i,e)} = u^{(i,e)}/U, \quad U = \frac{gR_0^2(^{(e)} - {}^{(i)})}{8\mu^{(i)}}, \qquad (21)$$

leading to the long time scale ${}^{-1/2}T$ for vertical dynamics where

$$\tilde{t} = {}^{1/2}t/T, \quad T = L/U. \qquad (22)$$

To nondimensionalize the pressure, the characteristic scale is chosen so that the vertical pressure gradient within the conduit balances the viscous force due to radial variation in the vertical velocity,

$$\tilde{p}^{(i,e)} = {}^{1/2}\frac{p^{(i,e)} - p_0}{}, \qquad = \mu^{(i)}U/L. \qquad (23)$$

Like in dimensional variables, the nondimensional, modified pressure can be decomposed as $\tilde{p}^{(i,e)} = \tilde{P}^{(i,e)} - \tilde{p}_h^{(i,e)}$, where $\tilde{P}^{(i,e)} = {}^{1/2}P^{(i,e)}/$ is the scaled, absolute pressure and $\tilde{p}_h^{(i,e)}$ is the normalized hydrostatic pressure which takes the form

$$\tilde{p}_h^{(i,e)} = -{}^{1/2}\frac{^{(i,e)}gz}{} = \frac{-{}^{(i,e)}\tilde{z}}{^{(e)} - {}^{(i)}}. \qquad (24)$$

Surface tension was neglected in the discussion of the uniform conduit, but it will be included in the full system of equations for completeness, so it is normalized about a characteristic scale :

$$\tilde{} = / . \qquad (25)$$

The Reynolds numbers for the viscous fluid conduit system are therefore defined for the two fluids according to

$$Re^{(i,e)}$$

derive information about higher order corrections in special cases. In what follows, we determine the scalings such that all corrections to the conduit equation (1) are O( ).

### A. Viscous, higher order corrections

The equations solved in deriving the conduit equation (1) in Sec. III B were a special case of the StokesÕ ßow equations, in which the vertical dynamics occurred over a much longer length scale than the radial dynamics. A convenient analytical property of the axisymmetric StokesÕ ßow equations, is that one can rewrite the nondimensional equations in the form [23]

$$\tilde{\nabla}^2 p^{(i,e)} = \frac{1}{r}\frac{}{r}\left( r\frac{p^{(i,e)}}{r}\right) + \frac{{}^2 p^{(i,e)}}{z^2} = 0, \qquad (58)$$

$$\tilde{\mathcal{L}}^2 \;{}^{(i,e)} = 0, \quad \tilde{\mathcal{L}} = \frac{{}^2}{z^2} + \frac{{}^2}{r^2} - \frac{1}{r}\frac{}{r}, \qquad (59)$$

where ${}^{(i,e)}$ is the StokesÕ stream function, which is related to the velocity components by

$$u_r^{(i,e)} = -\;{}^{1/2}\frac{1}{r}\frac{{}^{(i,e)}}{z}, \quad u_z^{(i,e)} = \frac{1}{r}\frac{{}^{(i,e)}}{r}. \qquad (60)$$

In the asymptotic formulation, the ßuid pressures and velocities were expanded in asymptotic series and expressions for the leading order term in the expansion were found. It was unclear

FIG. 2. (Color online) Figure[redacted] vertically uniform, intrusive cond[redacted] then perturbed by a steplike incre[redacted] One can see the formation of a D[redacted] wavetrain slowly modulated over [redacted] interesting observation is that the [redacted] within individual waves of the DSW[redacted] ßuid.

and Whitehead [5] observed a simila[redacted] the amplitude of conduit waves exce[redacted] the small slope condition $^{1/2}$R 1 in[redacted] evaluation gives $^{1/2}$R 4.5 and 2, respe[redacted] indeed the perturbed conduit radius ex[redacted] threshold and inertial effects would need to [redacted] model the conduit dynamics accurately for l[redacted] Hence our criteria accurately predict the point[redacted] of the conduit equation as an approximate desc[redacted] interface.

[redacted] With the limits of validity of the theoretic[redacted]blished, viscous ßuid conduits provide an [redacted]r the precise, quantitative, experimental st[redacted]ispersively regularized shock waves have a[redacted]eal of interest in recent years due to their c[redacted]ange of physical systems, to include ultraco[redacted]26,27], ion-acoustic plasma [28], nonlinear op[redacted]nd shallow water [31], but careful compariso[redacted]nd data are lacking. One difÞculty is the lo[redacted]me scales required for the study of DSWs[redacted]odulated wavetrains are characterized by [redacted]o scales. One is the O(1) scale of individual os[redacted]e other is a long, slow scale of wave modu[redacted] 1 (generally, ÷is different from deÞned[redacted]wever, images from previous experiments c[redacted] experimental study of DSWs is access[redacted]d conduits, e.g., Fig. 2. In this setting, a DS[redacted] steplike increase in the injection rate. [redacted]ger trailing, vertically uniform conduit c[redacted]er, leading vertically uniform conduit b[redacted]ting conduit waves, as depicted in Fig. 2[redacted]tomated syringe pump, high resolutio[redacted]e measurement of the ßuid densities[redacted] quantitative experiments are possible[redacted]dity of the conduit equation (1), meas[redacted]stic DSW featuresÑleading and traili[redacted]g edge amplitudeÑcan be comp[redacted]sults of asymptotic modulation the[redacted]r the conduit equation (1) by the pre[redacted]

[redacted] equation is asymptotically [redacted]s in the small amplitude, lo[redacted]SW regimes, e.g., backß[redacted]n observed in large ampl[redacted]work shows that these f[redacted]c features of the red[redacted] in viscous ßuid con[redacted]

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