Spatial patterns of desynchronization bursts in networks

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other parts of the network substantially synchronized. (This a somewhat counterintuitive effect related to the fact that, in some cases, increasing the coupling strength destabilizes the synchronous state [19,28].)

Arbitrarily small amounts of mismatch will eventually, through the bubbling mechanism, induce desynchronization bursts. We will show that some of the spatial patterns of this possibly microscopic mismatch might get amplified to a macroscopic size in the bursts. We will discuss how one can use knowledge of the parameter mismatch of the dynamical units in the network to decrease the effective size of the mismatch driving the bursts, thereby improving the robustness of the synchronization.

If synchronization is desired, the network and the parameters should be constructed so that the synchronous state for the identical oscillator system is robustly stable (this implies the absence of noise or mismatch induced desynchronization bursts). Even then, the synchronization will not be perfect if the oscillators have parameter mismatch. We will describe the characteristics of the deviations from exact synchronization in terms of the mismatch and the master stability function.

$$\hbar = [DF(s) -$$

our example, this region corresponds to 0.16 , a , 0.48 or 3.8 , a , 4.5. The range 0.48 , a , 3.8 will be referred to as the *stable region*, and the remaining zone will be called the *unstable region*.

this time, the trajectory closely follows the period 1 orbit, which is the most transversally unstable of the periodic orbits. Similar observations have been previously reported for two coupled chaotic systems [27]

difference $x_5 - x_4$ increases approximately at $t = 9000$ and returns to a relatively small value after reaching values considerably above the average.

In Fig. 11(a) we plot the difference between the x coordinate of node j and its mean over all nodes, $x_j - \overline{x}$, where $\overline{x} = 1/N$

vector, the strength of the mismatch affecting the localized mode is proportional to the weight of the localized eigenvector in the eigenvector decomposition of the mismatch. We will now discuss two applications of these results.

A. Amplification of mismatch patterns when modes with the same eigenvalue burst

We have shown that the modes of the mismatch force the corresponding modes of the deviations from the synchronous state. When bubbling induced bursting is expected, the size of the mismatch determines the average time between bursts [25]. Thus, the size of the mismatch component in mode k determines the average interburst time when that mode is in the bubbling regime.

When the spectrum of the matrix G is degenerate, the spatial modes of the mismatch play an extra role. All the modes sharing the same eigenvalue λ have the same stability properties, and thus, when the corresponding value $\gamma\lambda$ is in the bubbling zone, all eigenvectors with this eigenvalue are equally likely to appear. The only difference between these modes is the strength with which they are forced, which is determined by the mismatch component in that mode as shown in Eq. (16) (or, if noise is present, by the noise component in that mode).

An example of this situation is the ring with connections of equal strength in the long wavelength bursting scenario. Since the ring is invariant with respect to rotations, the phase of the long wavelength oscillations can not be determined only from the network and dynamics part of the problem. The two modes with the longest wavelength (corresponding to sinusoidal and cosinusoidal oscillations) have the same eigenvalue. It is the mismatch that in this case determines the phase of the long wavelength burst.

We will show how one can determine the phase of the long wavelength desynchronization burst in the case of coupled Rössler systems in a ring with equal coupling along each link. For this system, the mismatch vector $Q_j(X_j)$ is given by

$$Q_j([x_j,y_j,z_j]^T) = \begin{pmatrix} 0 \\ y_j\delta a_j \\ \delta b_j - z_j\delta c_j \end{pmatrix}, \tag{17}$$

where $\delta a_j = a_j - \overline{a}$ and similarly for δb_j and δc_j. We define $\mathcal{F}_k(u) = \sum_{j=1}^{N} u_j \hat{w}_j^k$, where $\hat{w}_j^k$ is the normalized jth component of the k eigenvector described at the beginning of Sec. III. With this convention, the term $(QL)_k$ in Eq. (16) is given by

$$(QL)_k = \begin{pmatrix} 0 \\ y\mathcal{F}_k(\delta a) \\ \mathcal{F}_k(\delta b) - z\mathcal{F}_k(\delta c) \end{pmatrix}. \tag{18}$$

Here $\delta a = [\delta a_1, \delta_2, \ldots, \delta_N]$ and similarly for δb, δc, and y, z are the trajectories around which the linearization was made.

We consider the case in which mismatch in one parameter is dominant, for example a. The mismatch in the parameters b and c will be assumed negligible compared with that in a, so that $\delta b, \delta c \ll \delta a$. In this case, only the second component

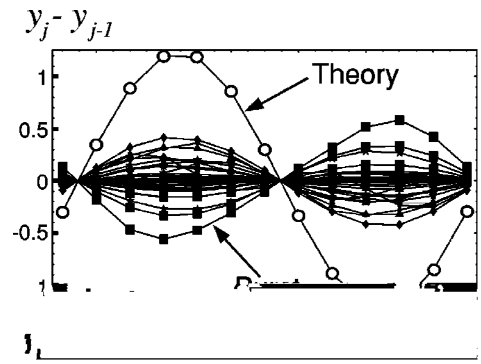

FIG. 13. $y_j - y_{j-1}$ for different times during a burst (filled symbols), and a scaled version of $\sin[(2\pi j/12)+\phi] - \sin([2\pi(j-1)/12]+\phi)$ with ϕ as given in the text (open circles). The phase of the burst spatial pattern coincides with the phase of the long wavelength component of the mismatch.

of Eq. (18) is of relevance. Thus modes η_1 and η_{N-1} are excited with a strength proportional, respectively, to $\mathcal{F}_1(\delta a)$ and $\mathcal{F}_{N-1}(\delta a)$; see Eq. (16). The magnitude of η_k will be proportional to $\mathcal{F}_k(\delta a)$, and thus the excitation of the long wavelength mode (which is the only one for which perturbations grow) is proportional to

$$\mathcal{F}_1(\delta a)\sin\left(\frac{2\pi j}{N}\right) + \mathcal{F}_{N-1}(\delta a)\cos\left(\frac{2\pi j}{N}\right) \tag{19}$$

$$\sim \sin\left(\frac{2\pi j}{N} + \phi\right), \tag{20}$$

where $\tan\phi = \mathcal{F}_{N-1}(\delta a)/\mathcal{F}_1(\delta a)$.

We now show results of numerical simulations illustrating the above. The parameters N and γ will be as in the long wavelength example in the preceding section. We use the same random set of perturbations used in that example. As described above, we obtained the phase ϕ of the long wavelength component of the vector δa_i. In Fig. 13 we plot $y_j - y_{j-1}$ for different times during a burst (filled symbols). In the same figure, we plot a scaled version of $\sin[(2\pi j/12) + \phi] - \sin([2\pi(j-1)/12] + \phi)$ (open circles). The phase of the desynchronization burst is in agreement with that of the long wavelength component of the mismatch.

When the mismatch affects predominantly one parameter as in this case, the phase of the bursts can be predicted as described above. When mismatch in different parameters is comparable, the phases of the long wavelength modes of the different parameter mismatches compete and the bursts develop with one of these phases or with a combination of them.

It must be emphasized that this analysis is possible only when there is a degeneracy of the eigenvalues. For example, the location of the localized burst can not be determined in this way, as it is fixed in the position of the strengthened link. In this case, the mismatch component in the localized mode would only affect the average time between bursts.

B. Artificial supression of unstable modes using knowledge of the mismatch

We will now discuss another consequence of Eq. (16). We imagine a situation where we are given a number of nearly identical oscillators that we are to connect in a network which we desire to be in synchronism as much as possible. Furthermore, we imagine that, through measurements made individually on each oscillator, we are aware of the amount of mismatch in each oscillator. The question we address is this: Using our knowledge of the individual mismatches, how should we arrange the oscillators in the network so as to best suppress the frequency of desynchronism bursts? To answer this question, we note that, according to the previous discussion, we should reduce the mismatch component in the mode which is in the bubbling region. Since the size of the mismatch affects the average interburst time [25], reducing this component is desirable if one wants to improve the quality of the synchronization. This can be accomplished by judiciously arranging the dynamical units so that the *k*th mode of the mismatch is minimized when the corresponding value $g\lambda_k$ is in the bubbling region. For example, to suppress long wavelength bursts, one may arrange the units so that the parameter errors alternate above and b72lthemion.48 65.4(T)69.9(-)]TJ T* [(suppress46357.6(the)6357.lomicimized)6357.0(burrding)635
ware44299.4inilarre44299.[(parametsre44299.aware44299.6(the)4299.7(osre44299.8(in44299.6(the)4299.8(regihe)4299.7fin44299.6(thg)]T

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dom perturbation to the parameter a of each oscillator chosen uniformly from within a ±0.1% range of $a=0.2$.

In Fig. 15 we show, for $k=1,\ldots,7$, the quantities $\langle|h_k|\rangle$ (squares), $\langle|(QL)_k|\rangle$ (triangles), and $\langle|(QL)_k|\rangle/|h_k|$ (circles).

The magnitudes of the forcing term for the different modes $[\langle|(QL)_k|\rangle]$ span roughly two orders of magnitude, and the magnitude of the response ($\langle|h_k|\rangle$) looks roughly proportional to the latter. When the forcing term is corrected by dividing it by the magnitude of the corresponding Lyapunov vector $|h_k|$, the resulting quantity $[\langle|(QL)_k|\rangle/|h_k|]$ matches very well the observed response.

VI. CONCLUSIONS

We have studied the stability properties of the synchronized state in a network of coupled chaotic dynamical units when these have a small heterogeneity. We have shown that when the dynamical units that are coupled in a network are slightly different, the synchronized state can be interrupted by large infrequent desynchronization bursts for some values of the parameters. The range of the parameters for which this phenomenon is expected can be obtained by performing a master stability function analysis of the chaotic attractor and of the periodic orbits embedded in it.

The desynchronization bursts are induced by the bubbling phenomenon, and have spatial patterns on the network. These spatial patterns can be predicted from the eigenvectors of the Laplacian matrix G and the master stability functions mentioned above. We showed examples illustrating the development of bursts with spatial patterns. One of our examples showed that the strengthening of a single connection might destabilize the nodes near this connection, while leaving the rest of the network approximately synchronized.

Direct measurement of the parameter mismatch in the elements of a network might prove useful. We discussed how this knowledge could be used to reduce the frequency of bursts and to predict the relative weights of different spatial patterns in a burst. We also discussed how one could, from knowledge of the mismatch and of the master stability function, describe the spatial patterns and magnitude of the deviations from the synchronized state when the synchronization of the corresponding identical unit system is robust.

We emphasize that although we did not discuss the effects of noise, the phenomenon described in this paper also occurs for noisy identical oscillators. Desynchronization bursts with spatial patterns are expected for noisy, identical oscillators if one has them for noiseless, nonidentical oscillators. The difference is that the parameter mismatch is always "frozen," in the sense that the mismatch is always the same for each oscillator, whereas for noise this is not the case.

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