Applied Analysis Prelim

10.00am–1.00pm, August 16, 2011

Problem 1. [Fixed point theorem] Show that the equation:

$$v(x) = \sin(x) + \int_0^x v^2(s)ds$$

has a solution in $C^1([0;\epsilon];R)$ for some $\epsilon > 0$.

Problem 2. Let X be the linear space of all functions $f : \mathbb{N} \to \mathbb{R}$, where $\mathbb{N} = \{1; 2; 3; \ldots\}$, such that

$$\|f\| := \sqrt{\sum_{k=1}^{\infty} (k+1) \, |f(k)|^2} < +\infty.$$

Show that $(X; \|\cdot\|)$ is a Banach space.

Problem 3. Show that the limit

$$\lim_{\epsilon \to 0^+} \int_0^{\infty} \frac{1 - e^{-\epsilon x}}{\epsilon} f^2(x)dx$$

exists and is finite if and only if $\int_0^{\infty} x f^2(x)\,dx < \infty$.

Problem 4. Let $k : [0;1] \times [0;1] \to \mathbb{R}$ be a continuous function and, for each $n \geq 1$, let $K_n : C[0;1] \to C[0;1]$ be the linear operator defined as

$$(K_n f)(x) := \frac{1}{n} \sum_{i=1}^{n} k\left(x; \frac{i}{n}\right) f\left(\frac{i}{n}\right).$$

(i) Show there exists and determine explicitly a bounded linear operator $K : C[0;1] \to C[0;1]$ such that $K_n \to K$ strongly.

(ii) Is it in general true or false that $K_n \to K$ uniformly? Justify your answer with a mathematical proof or a counter-example.

Problem 5. [True/False question - no justification] In this problem, $A : l^2 \to l^2$ is a self-adjoint compact operator. We define

$$E = \{\lambda \mid \lambda \text{ is an eigenvalue of } A\}.$$

Here we do not count multiplicity. Give a true or false answer to the following statements.

(1) Some of such an operator is invertible.

(2) For some A, $S \subset E$ with $S = \{1 - 1/n \mid n \in N;$ all positive intergers$\}$

(3) For some A, $E = \{1; 2; 3\}$

(4) For some A, $E = \{0; 1; -1\}$