10.00am–1.00pm, August 21, 2012

Instructions.

Solution sk tch s:

1. Integrating the di erential equation, we get

$$v(x) = \frac{1}{4} + \frac{1}{4}\int_0^{x} \sin(s + v^2(s))ds. \tag{1}$$

Let $|| \ ||_u = \sup_{\in[0,1]} |u(t)|$ denote the uniform norm, and define the set

$$X = \{\phi \in C[0,1] : \phi(0) = \frac{1}{4} \text{ and } ||\phi||_u \leq 1\}.$$

The set X combined with the uniform norm is a metric space. Now define the operator

$$[T\phi](x) = \frac{1}{4} + \frac{1}{4}\int_0^{x} \sin(s + (\phi(s))^2)ds;$$

the IVP can then be written as a fixed point problem $Tv = v$.

First observe that if $\phi \in X$, then $T\phi$

· Let I denote the line in the complex plane $I = \{z \in \mathbb{C} : \operatorname{Im}(z) = 0 \ \operatorname{Re}(z) \in [-\frac{\pi}{2}; \frac{\pi}{2}]\}$.

(a) Set $\alpha = \beta + i\gamma$ where β and γ are real. Set $C = \sup_{, \in I} |\alpha + z| = \sqrt{(\frac{\pi}{2} + |\beta|)^2 + \gamma^2}$. Since $|[Au](x)| \leq C|u(x)|$ for all x, we get $||A|| \leq C$. For the converse, suppose that $\beta \geq 0$ (the proof for $\beta < 0$ is analogous). Set $u \ = \chi_{[\ , \ +1]}$. Then $||u \ || = 1$ and

$$||Au \ ||^2 = \int^{+1} |(\alpha + \arctan(x))|^2 \, dx = \int^{+1} \left((\beta + \arctan(x))^2 + \gamma^2 \right) dx \geq (\beta + \arctan(n))^2 + \gamma^2 \to C:$$

(b) We have

$$(Au; v) = \bar{\alpha} \int_{\mathbb{R}} \overline{u(x)}\, v(x)\, dx + \int_{\mathbb{R}} \arctan(x)\, \overline{u(x)}\, v(x)\, dx \tag{2}$$

$$(u; Av) = \alpha \int_{\mathbb{R}} \overline{u(x)}\, v(x)\, dx + \int_{\mathbb{R}} \arctan(x)\, \overline{u(x)}\, v(x)\, dx: \tag{3}$$

We see that A is self-adjoint if and only if α is real.

(c) Suppose that $Au = 0$. Then $(\alpha + \arctan(x))\, u(x) = 0$ almost everywhere. This can happen only if $u = 0$. It follows that A is one-to-one for all α.

(d) If $\alpha \notin I$, then set $\delta = \min_{, \in I} |\alpha - z| = \operatorname{dist}(I; \alpha)$. Since I is closed, $\delta > 0$. Clearly $||Au|| \geq \delta ||u||$, so A has closed range. To prove the converse, we will use that since A is one-to-one for all α, it has closed range if and only if it has a continuous inverse. Suppose first that $\alpha \in (-\pi{=}2; \pi{=}2)$. Set $I \ = (\tan(\alpha) - 1{=}n; \tan(\alpha) + 1{=}n)$ and $u \ = \chi_{I_n}$. Then $\lim_{\ \to 0} ||Au \ ||{=}||u \ || = 0$, so A does not have a bounded inverse. If $\alpha = \pm\pi$, then use $u \ = \chi_{\pm[\ , \ +1]}$ to show that A is

To prove the statement about the sum, we di erentiate f to find

$$f'(t) = -\sum_{=1} (-1) \; e^{-} \quad = -\sum_{=1} (-e^{-}) \quad = -\frac{(-e^{-}) - (-e^{-})^{+1}}{1 - (-e^{-})} = \frac{1}{e + 1} + \frac{(-1)^{+1} e^{-}}{e + 1}.$$

Since $\lim_{\to\infty} f(t) = 0$, we have

$$f(t) = -\int^{\infty} f'(s)\, ds = -\int^{\infty} \frac{1}{e + 1}\, ds + (-1) \int^{\infty} \frac{e^{-}}{e + 1}\, ds.$$

The absolute value of the integrand in the second term is bounded by the L^1 function $g(t) = (e + 1)^{-1}$. We can therefore invoke dominated convergence as $N \to \infty$ to establish that the second term converges to zero.