Applied Analysis Preliminary Exam
1:30 PM – 4:30 PM, January 9, 2020

Instructions — You have three hours to complete this exam. Work all five problems. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are directly proving such a theorem (when in doubt, ask the proctor). Write your student number on your exam, not your name. Each problem is worth 20 points. (There are no optional problems.)

1. The following two problems are unrelated.
   (a) Let X and Y be normed vector spaces, and $D \subseteq X$ a convex subset of X. If $f : D \to Y$ is Hölder continuous with exponent $\alpha > 1$, prove that in fact f is a constant function.
   Hint: recall Hölder continuity with exponent α means $\|f(x) - f(x')\|_Y \le c\,\|x - x'\|_X^{\alpha}$ for some constant c, for all $x, x' \in D$.
   Hint: you may wish to apply the triangle inequality repeatedly.
   (b) Consider the following variant of the Weierstrass Function: $f(x) = 2\sum_{n=1}^{\infty} a^n \cos(b^n x)$ with $a = \frac{1}{8}$ and $b = 49$ (this function is not differentiable at any point). Prove
      i. that $f \in L^2(\mathbb{T})$ where $\mathbb{T}$ is the 1D torus of length 2π, and
      ii. that f is continuous.

2. Let $I = [0,1]$ and $k : I^2 \to \mathbb{R}$ be a continuous function. Fix some $1 \le p \le \infty$, and define

$$\forall f \in L^p(I),\ \forall x \in I, \quad \big(Tf\big)(x) = \int_0^1 k(x,y) f(y)\, dy.$$

   (a) Prove that Tf is a continuous function on I.
   (b) Prove the image of the unit ball in $L^p(I)$ is pre-compact in $C(I)$

3. Let $A \in B(H)$ be a bounded linear operator on a Hilbert space.
   (a) Let $\lambda \neq 0$ be in the point spectrum of A, and define the corresponding eigenspace M to be the set of all associated eigenvectors. Prove M is a Hilbert space.
   (b) If we also assume that A is a compact operator, prove that M must be finite dimensional

4. Let X be a normed linear space. If $x_n \rightharpoonup x$ in X, prove $\|x\| \le \liminf \|x_n\|$.

5. If $f \ge 0$ is measurable and $\int_E f\, d\mu = 0$ (where μ is the Lebesgue measure), prove that $f(x) = 0$ almost everywhere on E.
   Hint: define $E_n = \{x \mid f(x) > 1/n\}$ and consider $\bigcup_{n=1}^{\infty} E_n$. Note that measures are countably sub-additive.