Department of pplied Mathematics
reliminary Examination in Numerical nalysis
ugust, 2013

January 29, 2014

Solutions:

1. **Root Finding.**
   We ant to find a function such that the e ton iterations $x_{n+1} = x_n \quad f(x_n)=f^0(x_n)$ 'hop about' forever ithin a finite interval, ithout ever converging. The easiest example ould seem to be if the iterates form some short c cle, the simplest of all such arising if $x_{n+1} = \quad x_n$, i.e. $\quad x_n = x_n \quad f(x_n)=f^0(x_n)$. Simplif ing the notation b riting x in place of x_n, this ill be satisfied if $f^0(x) = \frac{1}{2x} f(x)$. We can thus choose $f(x) = \begin{cases} c\sqrt{x} & \text{if } x \quad 0 \\ c\sqrt{\quad x} & \text{if } x < 0 \end{cases}$, here c is an arbitrar constant.

2. **Numerical uadrature.**

   (a) Let h denote the length of a single subinterval before the extrapolation is done. Including also the subinterval midpoint, the trapezoidal rule over this subinterval ould have the eights at its ends and midpoint: $T_0 = h[\frac{1}{2}\ 0\ \frac{1}{2}]$ and, hen using also the midpoint $T_1 = h[\frac{1}{4}\ \frac{1}{2}\ \frac{1}{}$

3. **Interpolation/Approximation.**
We start b multipl ing the numerator and denominator of $p_n(x)$ b $\omega_n(x)$, to obtain

$$p_n(x) = \frac{\sum_{j=0}^{n} w_j f(x_j)(x - x_0) \cdots (x - x_{j-1})(x - x_{j+1}) \cdots (x - x_n)}{\sum_{j=0}^{n} w_j (x - x_0) \cdots (x - x_{j-1})(x - x_{j+1}) \cdots (x - x_n)}.$$

ote next that $\omega'(x)$ ill become a sum of $n+1$ terms, all but one vanishing hen substituting $x \to x_j$. Hence,

$$\omega'(x_j) = (x_j - x_0) \cdots (x_j - x_{j-1})(x_j - x_{j+1}) \cdots (x_j - x_n).$$

Substituting $w_j = 1/\omega'(x_j)$ into the expression for $p_n(x)$ above thus gives

$$p_n(x) = \frac{\sum_{j=0}^{n} f(x_j) \frac{(x-x_0)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_n)}{(x_j-x_0)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_n)}}{\sum_{j=0}^{n} 1 \frac{(x-x_0)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_n)}{(x_j-x_0)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_n)}}.$$

The denominator is identicall one (being the Lagrange interpolation pol nomial to data that is one at ever node point), and it can therefore be omitted. The expression has then reduced to the standard form of the Lagrange interpolation pol nomial.

4. **Linear algebra**

   (a) Reduction to Hessenberg form can be performed using either Housholder reflections or Givens rotations. The Hessenberg form of a general matrix is of the form

$$\begin{pmatrix} x & x & \cdots & x & x \\ x & x & \cdots & x & x \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & x & x \\ 0 & 0 & \cdots & x & x \end{pmatrix}$$

   and is tridiagonal for self-adjoint matrices. Its purpose is to reduce the computational burden of QR iteration.

   (b) Initialize $\mathsf{A}_1 = \mathsf{A}$. QR iteration proceeds as

$$\begin{aligned} \mathsf{A}_n &= \mathsf{Q}_n \mathsf{R}_n \\ \mathsf{A}_{n+1} &= \mathsf{R}_n \mathsf{Q}_n, \quad n = 1, 2, \ldots \end{aligned}$$

   for $n = 1, 2, \ldots$. Here Q_n is unitar and R_n is upper triangular. These matrices are obtained via QR factorization. The second step is the product of these matrices in the reverse order.

   (c) We have

5. **ODEs**

(a) We have

$$\mathbf{f}(t_n + h; \mathbf{y}_n + \mathbf{k}_1) = \mathbf{f}(t_n; \mathbf{y}_n) + h \left(\frac{\partial \mathbf{f}(t_n; \mathbf{y}_n)}{\partial t} + \frac{\partial \mathbf{f}(t_n; \mathbf{y}_n)}{\partial \mathbf{y}} \mathbf{f}(t_n; \mathbf{y}_n) \right) + O(h^2); y_n + \mathbf{k}$$

. **PDEs**

We look for the solution in the form

$$u(x;y) = \sum_{m=0}^{M\ 1}\sum_{n=0}^{N\ 1} u_{mn} \sin(\ (m+1)\,x) \sin\ \ n + \frac{1}{2}\ \ y$$

so that

$$\frac{@u}{@y}(x;y) = \sum_{m=0}^{M\ 1}\sum_{n=0}^{N\ 1} u_{mn}\ \ n + \frac{1}{2}\ \ \sin(\ (m+1)\,x) \cos\ \ n + \frac{1}{2}\ \ y$$

satisfies the eumann boundar at $y = 1$. Computing

$$u(x;y) = \sum_{m=0}^{M\ 1}\sum_{n=0}^{N\ 1} u_{mn}\ {}^{2}\ (m+1)^2 + \ n + \frac{1}{2}\ {}^{2}\ ! \sin(\ (m+1)\,x) \sin\ \ n + \frac{1}{2}\ \ y\ \ ;$$

e seek an expansion of the right hand side,

$$f(x;y) = \sum_{m=0}^{M\ 1}\sum_{n=0}^{N\ 1} f_{mn} \sin(\ (m+1)\,x) \sin\ \ n + \frac{1}{2}\ \ y$$

so that e can set

$$u_{mn} = \frac{f_{mn}}{{}^{2}\ (m+1)^2 + \ n + \frac{1}{2}\ {}^{2}} :$$

Consider $x_k = (k+1) = M$, $k = 0;:::;M\ \ 1$ and $y_l = (l+1) = N$, $l = 0;:::;N\ \ 1$ so that

$$f(x_k;y_l) = \sum_{m=0}^{M\ 1}\sum_{n=0}^{N\ 1} f_{mn} \sin\ \frac{(m+1)\,k}{}$$

1f
94761F40 010 1978 7867818 964539.8999 emcjh0[]0 0 d 86fJ 16.436 N1 w T0f 0 7.80540234 Tf0 -1.291 TJ/TF1542910699701 TTf f 5378.818 [44(n