adjacency matrix is a key quantity in determining the onset of synchronization. Finally, we quantify the maximum achievable synchrony for a given network structure and find that this maximum value depends only weakly on the heterogeneity of the network's degree distribution.

The rest of the paper is organized as follows. After describing the model and its governing dynamical equations in Sec. II, we discuss the linear stability of the incoherent solution in Sec. III. We use this analysis to find the critical value of the coupling constant for the onset of synchronization. We then study the synchronized state in Sec. IV, obtaining a set of self-consistent equations for the local order parameters that determines the global order parameter as a function of the coupling strength. Approximate solutions of this set are obtained near the onset of synchrony and in the strong coupling limit. Finally, we discuss our results in Sec. V.

II. NETWORK HMF MODEL

In the original HMF model, and in most subsequent studies [1–4], the rotors in (1) were assumed to have the same moments of inertia, $I_n = 1$, and the coupling was assumed to be all-to-all with equal strength, $A_{mn} = 1$. While such a simplified setting provides many insights, interactions are rarely uniform and all-to-all in practice. For example, the HMF model is a simplified model for an n-body gravitational system in one spatial dimension with periodic boundary conditions, keeping only one harmonic of the potential [6,8]; in this case, the interaction strength should be proportional to the product of the particle masses and decay with the separation of the particles.

With this motivation we allow for a general adjacency matrix, A, in (1), but simplify by setting $I_n = 1$. The resulting dynamical system is

$$\dot{\theta}_n = p_n, \tag{2}$$

$$\dot{p}_n = \frac{K}{N}\sum_{m=1}^{N} A_{nm}\sin(\theta_m - \theta_n). \tag{3}$$

As is usual, it is convenient to define order parameters to quantify synchronization. When the network is heterogeneous, one can define a set of real, local order and phase parameters, $\{(R_n, \phi_n) : n = 1, \ldots N\}$, by

$$R_n e^{i\phi_n} = \frac{1}{N}\sum_{m=1}^{N} A_{nm} e^{i\theta_m}, \tag{4}$$

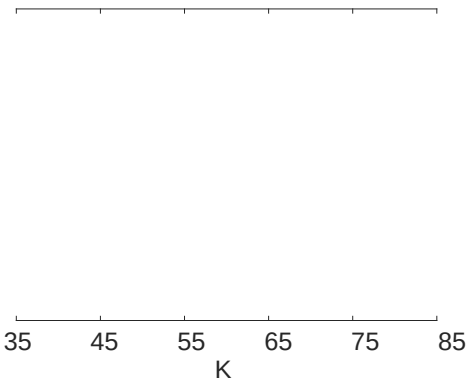
that characterize the coherence of inputs to a given node. Using these, (3) becomes

$$\dot{p}_n = K R_n \sin(\phi_n - \theta_n). \tag{5}$$

The overall synchrony of rotors can be measured by a global order parameter [17]

$$R = \frac{1}{d}\sum_{n=1}^{N} R_n. \tag{6}$$

Here $\ldots$ denotes the average over nodes,


35
45
55
65
75
85
K

average:

$$f_t \to f_{g_B} \equiv \int_0^{2\pi}\int_{-\infty}^{\infty} f(\theta_n, p_{\theta_n}) g_B(\theta_n, p_{\theta_n}; r_n)\, dp_{\theta_n}\, d\theta_n. \tag{23}$$

Here g_B is the Boltzmann distribution for the single-rotor energy

$$g_B(\theta, p_\theta, r) = \frac{1}{(2\pi)^{3/2} I_0(Kr)} e^{-(p_\theta^2/2 - Kr\cos(\theta))}, \tag{24}$$

for an inverse temperature that must be determined. The Bessel function, I_0, in the denominator normalizes the distribution: $\int_0^{2\pi}\int_{-\infty}^{\infty} g_B(\theta, p_\theta, r)\, dp_\theta\, d\theta = 1$. For this distribution, the mean-square momentum (in this case, the variance) is

$$\sigma^2 = \int_{-\infty}^{\infty}\int_0^{2\pi} p_\theta^2 g_B(p_\theta, \theta, r)\, d\theta\, dp_\theta = 1, \tag{25}$$

and the mean potential energy is proportional to

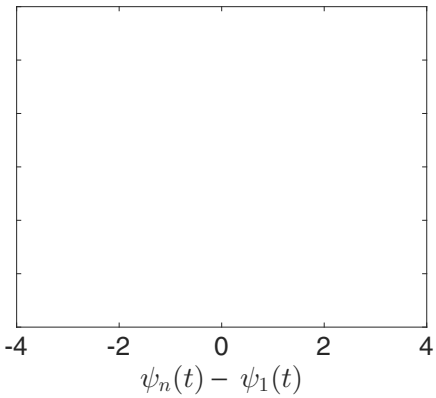
$$\int_{-\infty}^{\infty}\int_0^{2\pi} \cos(\theta) g_B(\theta, p_\theta, r)\, d\theta\, dp_\theta = v(Kr), \tag{26}$$

where we introduce the notation

$$v(x) \equiv \frac{I_1(x)}{I_0(x)}, \tag{27}$$

and I_1 is the first-order Bessel function.

Using (19) and (20) in the definition (4) of the local order parameter, we can solve for r


-4
-2
0
2
4
$\psi_n(t) - \psi_1(t)$

where the superscripts indicate the order in the perturbative expansion and we have anticipated already that $r_n \sim (K - K_c)^{1/2}$ (see, however, the last paragraph of this section). We have included only terms up to the order necessary to determine $r_n^{(1)}$ in the analysis that follows. Inserting these [34] and expanding in powers of ϵ we obtain at zeroth order,

$$\mu^{(0)} = \omega_0^2,$$

as expected. The next terms, of order $\epsilon^{1/2}$, imply

$$\left(A - \frac{K_c}{2\omega_0^2 N} I \right) r^{(1)} = 0,$$

which gives

$$K_c = \frac{2\omega_0^2 N}{\lambda}, \quad r^{(1)} = Cu. \tag{36}$$

Here u and λ are the principal eigenvector and eigenvalue of A, and C is a normalization constant to be determined (as we will see, the product Cu does not depend on the normalization of u). We have assumed that the only unstable mode is the one corresponding to the principal eigenvector. The gap between the real parts of the two leading eigenvalues thus determines the range of validity of this dominant-mode perturbative approximation. This result is in agreement with the linear stability calculation of Sec. III [cf. (15)]. The terms of order ϵ^1 lead to $r^{(2)} \propto u$ (although this will not be used), and to

$$\mu^{(1)} = \frac{K_c^2 C^2}{2}$$

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