A draft of INRIA lectures, May 1991

Wavelets, Multiresolution Analysis and Fast Numerical Algorithms

Bey n

M po e o eq e o de N ope on o co p e e

$$p_j = \sum_{i \neq j} \frac{q_i q_j}{| \ i}$$

e e od y e e ed de ce fo ed c n p d . en eq on o
p e ne y e fo e co of n n e en y cond on n e of e e n
ce f n e d of n e d . ence o n e e e en ep e en on e e e ep
e en on of e de e n e e e en p e od c on

e n y e

II.1 Multiresolution analysis.

e e de n on of e e o on n y no on n od ced y Meye , nd M , c p e e e en

f ee n en e of e en o o of ene y e
o e ne e n e d of e e en e

$$\mathbf{V_n} \subset \quad \subset \mathbf{V} \subset \mathbf{V} \subset \mathbf{V} \quad \mathbf{V} \subset \mathbf{L} \; \mathbf{R^d}$$

n n e c e z on e p ce $\mathbf{V}$ n e d en on

II.2 The Haar basis

o on e e on y e p e of e e o on n y fy n De n on
Cond on e Co p n n o p e of e o e de e n e ec e nd p o de f p o o ype fo n e c e pe en on f d en $_{j;k}$ $^{j=}$ ^{j} —
$\in \mathbf{Z}$ fo ed y e d on nd n on of n e f nc on

$$\begin{cases} & \text{fo} \\ - & \text{fo} \quad \leq \\ & \text{e e e} \end{cases} \qquad 7$$

n c e e e e c c e c f nc on of e n e
o e c $_{j;k}$ $^{j=}$ ^{j} — $\in \mathbf{Z}$ e of $\mathbf{V_j}$ nd $_{j;k}$ $^{j=}$ ^{j} — $\in \mathbf{Z}$ e of $\mathbf{W_j}$

e deco po on of f nc on n o n o de N p oced e en N ^{n} p e of f nc on c y fo p c y e o of e of ed e e of f on n e of en ^{n}

$$_{k} = {}^{n=} \int_{-n k}^{-n k+)} f \; d$$

e o n coe c en

$$d_k^{j+} = \frac{}{\sqrt{}} \; {}^{j}_{k} - {}^{j}_{k}$$

nd e e

$$^{j+}_{k} = \frac{}{\sqrt{}} \; {}^{j}_{k} \quad {}^{j}_{k}$$

fo n — nd ^{n} ^{j} — e y o e e n e o e
of coe c en d_k^j $^{j}_{k}$ n eq e N — dd on nd N p c on
n o d en on e e e o n

o e nd d fo e econd de ned y e e of ee nd of f nc on ppo ed on q e j;k j;k′ y j;k j;k′ y nd j;k j;k′ y e e e c c e c f nc on of e n e nd j;k $:=$ j= j $-$ ep e en n n ope o n e d o e non nd d fo e e no o y eco e c e e

By con de n n n e ope o

$$\int f \quad := \quad y \; f \; y \; dy$$

nd e p nd n e ne n o d en on e nd fo C de on Zy nd nd p e do d e en ope o e dec y of en e f nc on of e d nce f o e d on f e n e e ep e en on n n e o n e ne e e c of ope o e en y n e o d on e ne e oo y f o e d on o e p e e ne y of C de on Zy nd ope o fy e e e

$$| \quad y | \leq \frac{}{| \; - y|}$$

$$|\,_{\mathbf{x}}^{\mathbf{M}} \quad y| \quad |\,_{\mathbf{y}}^{\mathbf{M}} \quad y| \leq \frac{C_{\mathbf{M}}}{| \; - y|^{\;+\mathbf{M}}}$$

fo e $M \geq$ Le $M :=$ n nd con de

$$^{\mathbf{j}}_{\mathbf{kk'}} := \int\int \quad y \quad _{\mathbf{j;k}} \quad _{\mathbf{j;k'}} \; y \; d \; dy$$

e e e e e d nce e een $| \; - \; '| \geq$ nce

$$\int _{\mathbf{j;k}} \quad d \; :=$$

e e

$| \quad ; \quad y \quad f \; r \quad \mathrm{x}$

e on e n n e dec y n c en o e co p n n p c c o e f e dec y nece y o e f nc on e e n n o en e n n o en e e pon e fo n n p c c o e con o n e con n n e co p e y e e of e f o

II.3 Orthonormal bases of compactly supported wavelets

e q e on of e e ence of e on n y oo f nc on of Cond on e no e ed n e con c on of e o ono e e ene z n f nc on y o e . , nd Meye . , e con de on y co p c y ppo ed e e n n o en con c ed y D ec e . , fo o n e o of Y Meye . , nd M . , o o of e e e n d fo o e c o ce of e e e f e e c o c o ce of e e y on y con de n fo d en on $d \geq$ e con c ed f o o e fo d . –

Le con de e e on n y fo **L R**[7] n f d d e [7] econd e o ol on y of { − }k Z p e

Z + Z +
k − − d − | | e ik d

nd e efo e

Z X
k − | | e ik d
l Z

nd

X
| | −
l Z

n e o n

X
|
l Z

no d c o o ee e . , . , . , on e c ed e ∈ Z e
e e { j;k — j= j — }k Z fo n o ono of Wj
e fo o n e D ec e , c c e ze ono e c po yno
on of c co e pond o e o ono e of co p c y ppo ed
e e n n o en

Lemma II.1 Any trigonometric polynomial solution of (2.26) is of the form

$$h \quad -\tfrac{1}{2} \quad e^{i} \quad i_M \quad e^{i}$$

where $M \geq$ **is the number of vanishing moments, and where is a polynomial, such that**

$$| \quad e^{i} \quad | \quad -P \quad n \quad \tfrac{1}{2} \quad n^{M} \quad \tfrac{1}{2} \quad \tfrac{1}{2} \mathrm{co}$$

where

$$P \quad y \quad - \sum_{k}^{k \quad M} \quad M - \quad ! \quad y^{k} \qquad e$$

and is an odd polynomial, such that

$$\leq P \quad y \quad y^{M} \quad \tfrac{1}{2} - y \quad \textbf{for} \quad \leq y \leq$$

and

$$\mathrm{p}_{y} \; h \; P \quad y \quad y^{M} \quad \tfrac{1}{2} - y \; i \quad M \quad)$$

e p oof of and

e e $^{j}_{k}$ nd d^{j}_{k} y e e ed pe od c eq ence e pe od n j Co p n

nd ed y e py d e e

$$
\begin{array}{ccccccc}
\{\ _{k}\} & \longrightarrow & \{\ _{k}\} & \longrightarrow & \{\ _{k}\} & \longrightarrow & \{\ _{k}\} \ \cdots \\
& \searrow & & \searrow & & \searrow & \\
& & \{d_{k}\} & & \{d & &
\end{array}
$$

e en de ne $f_{\mathbf{m}} := f_{\mathbf{m}} - \mathbf{m} \cdot f$ e e $\mathbf{m}$ c o en o $\langle f_{\mathbf{m}} \; \mathbf{M} \rangle :=$ fo
$:=$ M c e n e de ed o o on y o $\mathbf{M}$ y e con n e o

e e $\mathbf{V}_j^{M}$ en n e nd e p ce $\mathbf{W}_j^{M;}$ e o o on co p e en of $\mathbf{V}_j^{M;}$ n $\mathbf{V}_j^{M;}$

$$\mathbf{V}_j^{M;} = \mathbf{V}_j^{M;} \; M \; \mathbf{W}_j^{M;}$$

e p ce $\mathbf{W}^{M;}$ p nned y e o ono

$$\{ \; i \quad l \, y \quad i \quad l \, y \quad i \quad l \, y \quad = \quad M\}$$

e e $\{ \; _m\}_m^{m \; M}$ n o ono fo $\mathbf{V}^{M}$ c e e en on e M e e en n n o en e o on o e po yno $^i y^l$ = $M -$

e p ce $\mathbf{W}_j^{M;}$ p nned y d on nd n on of e f nc on of $\mathbf{W}^{M;}$ nd e of $\mathbf{L}$ con of e f nc on nd e o o de po yno $^i y^l$ = $M -$

e no e e o d en on e e eq e M d en co n on of one d en on f nc on e e M e n e of n n o en On e o e nd e o d en on o ned y n co p c y ppo ed e e eq e on y ee c co n on c p e e con c on of e non nd d fo e ec on

II.5 A remark on computing in the wavelet bases

n y e no e once e e een c o en co p e e y de e ne e f nc on nd nd e efo e e e on n y n n e e n o e on n p ope y con c ed o e f nc on nd e ne e co p ed D e o e ec e de n on of e e e e n p on e pe fo ed e q d e o e nd e en f ey n o e q n e c ed nd A n e p e e co p e e o en of e n f nc on e e p e on fo e o en

$$\mathcal{M}^m = \int {}^m \; d \; = \; M -$$

n e of e e coe c en $\{ \; _k\}_k^{k \; L}$ y e fo nd n fo fo

$$= \prod_j \; {}_j$$

e e

$$= \sum_k {}^k \; {}_k e^{ik}$$

e o en $\mathcal{M}^m$ e o ned n e c y n e de ed cc cy y
ec e y ene n eq ence of ec o $\{\mathcal{M}^m_r\}^M_m$ fo $r =$

$$\mathcal{M}^m_{r+} = \sum_j^m \; j r \mathcal{M}^m_r \; j \mathcal{M}^j$$

n

$$\mathcal{M}^m = m^{\frac{1}{2}} \sum_k^k \; k^m = \; M -$$

c ec o $\{\mathcal{M}^m_r\}^M_m$ ep e en M o en of e p od c n r e
nd e on con e e p d y No ce e ne e co p ed e f nc on
e f

e non nd d nd nd d fo

III.1 The Non-Standard Form

Le e n ope o

$$\mathbf{L} \quad \mathbf{R} \rightarrow \mathbf{L} \quad \mathbf{R}$$

e e ne y De n n p o ec on ope o on e p ce $\mathbf{V_j}$ $\in \mathbf{Z}$

$$P_{\mathbf{j}} \quad \mathbf{L} \quad \mathbf{R} \rightarrow \mathbf{V_j}$$

$$P_{\mathbf{j}} f \quad = \sum_{\mathbf{k}} \langle f \quad {}_{\mathbf{j;k}} \rangle \quad {}_{\mathbf{j;k}}$$

nd e p nd n n e e op c e e e o n

$$= \sum_{\mathbf{j} \quad \mathbf{Z}} \quad \mathbf{j} \quad \mathbf{j} \quad \mathbf{j} \quad P_{\mathbf{j}} \quad P_{\mathbf{j}} \quad \mathbf{j}$$

e e

$$\mathbf{j} = P_{\mathbf{j}} \quad - P_{\mathbf{j}}$$

e p o ec on ope o on e p ce $\mathbf{W_j}$ f e e e co e n en n e d of e e e

$$= \sum_{\mathbf{j}} \quad \mathbf{j} \quad \mathbf{j} \quad \mathbf{j} \quad P_{\mathbf{j}} \quad P_{\mathbf{j}} \quad \mathbf{j} \quad P_{\mathbf{n}} \quad P_{\mathbf{n}}$$

nd f e e = e ne e en

$$= \sum_{\mathbf{j}} \quad \mathbf{j} \quad \mathbf{j} \quad \mathbf{j} \quad P_{\mathbf{j}} \quad P_{\mathbf{j}} \quad \mathbf{j} \quad P_{\mathbf{n}} \quad P_{\mathbf{n}}$$

e e $\sim$ $= P \quad P$ d e z on of e ope o on e ne e p n on e nd deco po e ope o n o of con on f o d en e

e non nd d fo ep e en on e of e ope o c n of p e

$$= \{A_{\mathbf{j}} \quad B_{\mathbf{j}} \quad {}_{\mathbf{j}}\}_{\mathbf{j} \quad \mathbf{Z}}$$

c n on e p ce $\mathbf{V_j}$ nd $\mathbf{W_j}$

$$A_{\mathbf{j}} \quad \mathbf{W_j} \rightarrow \mathbf{W_j}$$

$$B_{\mathbf{j}} \quad \mathbf{V_j} \rightarrow \mathbf{W_j}$$

$$\mathbf{W_j} \to \mathbf{V_j}$$

e e e ope o $\{A_j\ B_j\ \ _j\}_{j\ Z}$ e de ned A_j .— j j B_j .— j P_j nd $_j$.—P_j j

e ope o $\{A_j\ B_j\ \ _j\}_{j\ Z}$ d ec e de n on e e on

$$_j := \begin{pmatrix} A_{j+} & B_{j+} \\ _{j+} & _{j+} \end{pmatrix}$$

e e ope o $_j$.—P_j P_j

$$_j \quad \mathbf{V_j} \to \mathbf{V_j}$$

nd e ope o ep e en ed y e × n pp n

$$\begin{pmatrix} A_{j+} & B_{j+} \\ _{j+} & _{j+} \end{pmatrix} \quad \mathbf{W_{j+}} \oplus \mathbf{V_{j+}} \to \mathbf{W_{j+}} \oplus \mathbf{V_{j+}} \qquad (4)$$

f e e co e n en

$$.—\{\{A_j\ B_j\ \ _j\}_{j\ Z\ j\ n\ n}\}$$

e e $_n$.—P_n P_n f e n e of e n e en .— n n nd e ope o e o nzed oc of e ee. e nd

Le e e fo o n o e on

- e ope o A_j de e e n e c on on e e on y nce e p ce $\mathbf{W_j}$ n n e e en of e d ec n

- e ope o B_j $_j$ n nd de e e n e c on e een e e nd co e ndeed e p ce $\mathbf{V_j}$ con n e p ce $\mathbf{V_{j'}}$ $'$ ee

- e ope o $_j$ n e ed e on of e ope o $_j$

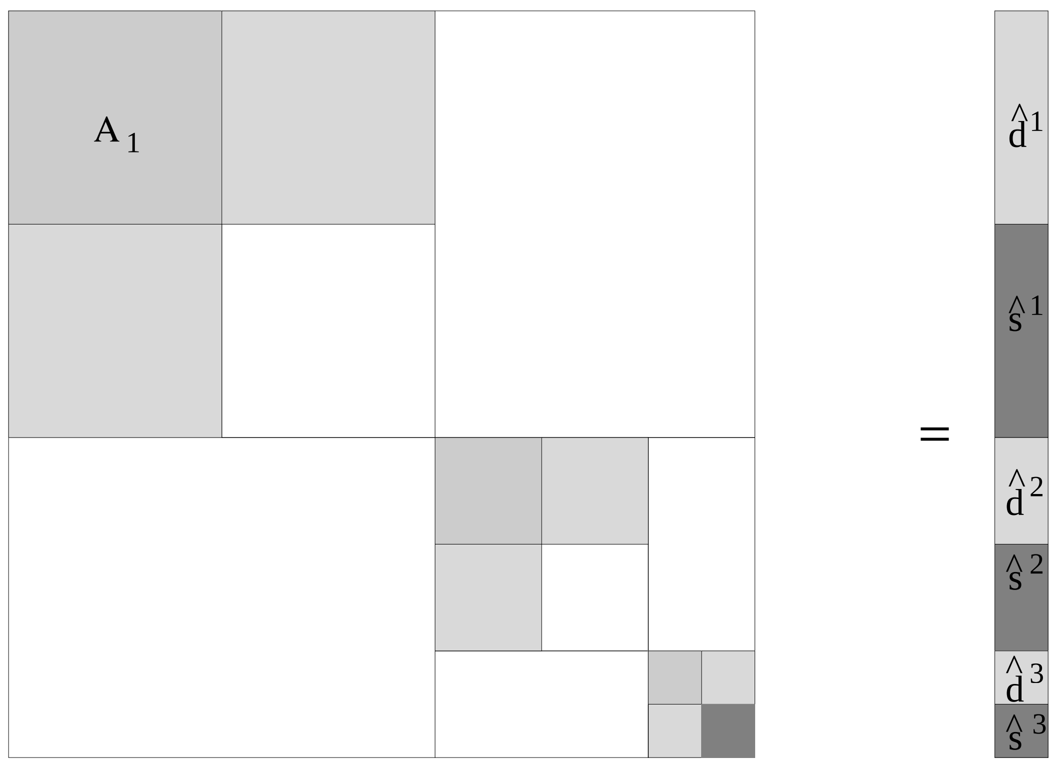
- e ope o A_j B_j nd $_j$ e ep e en ed y e ce $_j$ $_j$ nd $_j$

$$^j_{k;k'} := \int\int y \quad _{j;k} \quad _{j;k'}\ y\ d\ dy$$

$$^j_{k;k'} := \int\int y \quad _{j;k} \quad _{j;k'}\ y\ d\ dy \qquad (7)$$

nd

$$^j_{k;k'} := \int\int y \quad _{j;k} \quad _{j;k'}\ y\ d\ dy$$



A 1
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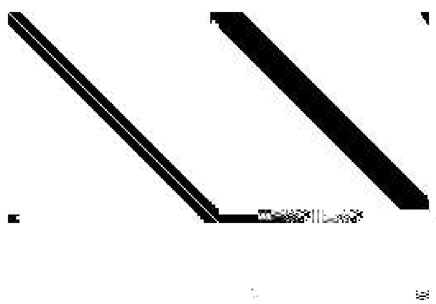
^s 2
^d 3
^s 3



e An e p e of n e non nd d fo ee p e

e ope o j ep e en ed y e j

$$ {}^{j}_{k;k'} = \int\int \quad y \quad {}_{j;k} \quad {}_{j;k'} \; y \; d \; dy $$

en e of coe c en ${}_{k;k'}$ $' =$ $N -$ epe ed pp c on of e [redacted]
fo 7 p od ce

[redacted]

$$ {}^{j}_{i;l} = \sum_{k;m}^{L} \quad {}_{k} \; {}_{m} \; {}^{j}_{k+\; i+\; ;m+\; l+} $$

III.2 The Standard Form

e nd d fo o ned y ep e en n

$$V_j = \bigoplus_{j'>j} W_{j'}$$

nd con de n fo e c e e ope o $\{B_j^{j'}, \Gamma_j^{j'}\}_{j'>j}$

$$B_j^{j'} : W_{j'} \to W_j$$

$$\Gamma_j^{j'} : W_j \to W_{j'}$$

7

f e e e co e n en n e d of e e

$$V_j = V_n \bigoplus_{j'=j+1}^{n} W_{j'}$$

n c e e ope o $\{B_j^{j'}, \Gamma_j^{j'}\}$ fo $j' =$ n e e e e n nd

7 nd n dd on fo e c e e e ope o $\{B_j^{n+1}\}$ nd $\{\Gamma_j^{n+1}\}$

$$B_j^{n+1} : V_n \to W_j$$

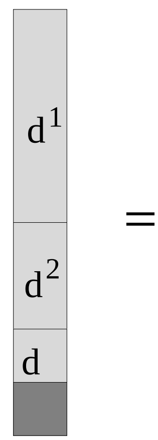
$$\Gamma_j^{n+1} : W_j \to V_n$$

n no on $\Gamma_n^{n+1} = \Gamma_n$ nd $B_n^{n+1} = B_n$ f e n e of e e n e nd V n e d en on en e nd d fo ep e en on of $= P$ P

$$= \{A_j, \{B_j^{j'}\}_{j'=j+1}^{n}, \{\Gamma_j^{j'}\}_{j'=j+1}^{n}, B_j^{n+1}, \Gamma_j^{n+1}\}_{j}$$ re of o C f

e ope o e o n zed oc of e ee e nd e e

f e ope o C de on Zy nd o p do eratoj /R36 (3aV)Tj /R360 Td (Figu65



d1
d2
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e co p e on of ope o

e co p e on of ope o o n o e o d e con c on of e p e ep e en on n o ono e d ec y ec e peed of co p on o e e co p e on of d of e fo e p e c e ed y e od o e n nd n p e ep e en on n e y e deq e fo e pp c on e co p e on of ope o c fo ep e en on n n o de o e ec e y co p e n e p e fo e nd d nd non nd d fo of ope o n e e e y e e ed co p e on e e e fo de c of nd non nd d fo of ope

the matrices ^{j}, ^{j}, ^{j} **(3.16) - (3.18) of the non-standard form satisfy the estimate**

$$| \,_{i;l}^{j}| \quad | \,_{i;l}^{j}| \quad | \,_{i;l}^{j}| \leq \frac{C_M}{| \; - \;|^{M+}}$$ e 7

for all $| \; - \;| \geq \; M$**.**

con de on pp y n e c e of p e do d . en ope o Le e p e do d . en ope o y o de ned y e fo

$$f \quad . - \; e^{ix} \int \quad f \quad d \; . - \int \quad y \; f \; y \; dy$$ e

e e e d on e ne of

Proposition IV.2 **If the wavelet basis has** M **vanishing moments, then for any pseudo-di erential operator with symbol of and of satisfying the standard conditions**

$$| \quad _x \quad | \leq C \; ; \quad | \; | \quad ^+$$ e

$$| \quad _x \quad | \leq C \; ; \quad | \; | \quad ^+$$ e

the matrices ^{j}, ^{j}, ^{j} **(3.16) - (3.18) of the non-standard form satisfy the estimate**

$$| \,_{i;l}^{j}| \quad | \,_{i;l}^{j}| \quad | \,_{i;l}^{j}| \leq \frac{^j \; C_M}{| \; - \;| \; ^{M+}}$$ e

for all integer , **.**

f e pp o e e ope o ^{N} y e ope o $^{N;B}$ o ned f o ^{N} y e n o ze o coe c en of ce $_{i;l}^{j}$ $_{i;l}^{j}$ nd $_{i;l}^{j}$ o de of nd of d $B \geq \; M$ o nd e d on en e y o ee

$$|| \; ^{N;B} - \; ^N || \leq \frac{C}{B^M} \; o \; N$$ e

e e C con n de e ned y e ene n o n e c pp c on e cc cy of c c on ed nd e p e e of e o n o c e e nd d B nd o de M e o e c o en n c nne e de ed p ec on of c c on c e ed f M ed en B o e c

$$|| \; ^{N;B} - \; ^N || \leq \frac{C}{B^M} \; o \; N \leq$$ e

y $-$ y (4.)

e e f c y n y n e f nc on e e nd e nd e efo e e ope o L nd L po e o dec de f C de on Zy nd ope o o nded

Theorem IV.1 (G. David, J.L. Journe) **Suppose that the operator (3.1) satis es the conditions (4.5), (4.6), and (4.16). Then a necessary and su cient condition for to be bounded on L is that in (4.24) and y in (4.25) belong to dyadic BMO, i.e. satisfy condition**

$$\mathrm{sup}_{J} \frac{1}{|J|} \int_{J} | \quad J \quad | \, d \leq C \qquad (4.)$$

where is a dyadic interval and

$$J \quad \frac{1}{|J|} \int_{J} \quad d \qquad (4.7)$$

p n e ope o n o e of ee e e nd e n e ep e y e d o e e e e e no e e f nc on nd e e y co p ed n e p oce of con c n e non nd d fo nd nd y e ed o p o de ef e e of e no of e ope o e d e en ope o n e e e

V.1 The operator d=dx in wavelet bases

e non nd d fo of e e co on y ed ope o y e co p ed e p c y e con c e non nd d fo of e ope o $d\ d$ e e e en of A_j B_j nd r of e e d d d

e e n e e oco e on coe c en of e e $\{ k \}_k^L$

$$n \quad L \sum_i^n \quad i \quad i+n \quad n \quad L$$

e y o ee e oco e on coe c en n e en nd ce e ze o

$$k \quad L$$

y e e fyed y n o co p e | nd |

$$| \quad \sum_n^L \quad n \; \cos n$$

$$| \quad \sum_k^{L} \quad k \; \cos \quad \sum_k^{L} \quad k \; \cos$$

e e n e en n Co n n nd o fy e o n

$$\sum_k^{L} \quad k \; \cos$$

nd ence nd

e e en o en of e coe c en k f o n n e y

$$\sum_k^{k \; L} \quad k \quad m \quad \text{fo} \quad \leq M$$

e

nce

$$m \quad | \quad \text{fo} \quad \leq M$$

c fo o f o e Co o d en f e 7 e d e y fy o 7 7 e f d cof e n e e e 7

$$r_i := \sum_{k}^{L} \sum_{m}^{k+} k \; k \; m \; r_{i+m}$$

C n n e o de of on n nd n e f c $\sum_{k}^{L}$ k e e

$$r_l := r_l \sum_{n}^{L} n \; r_l \; n \; r_{l+n} \in \mathbf{Z}$$

e e n e en n n e o n e f o n o de o o n e e fo o n e on

$$\sum_{l}^{l} m - := m \sum_{l}^{l\,m} - l \quad M_l' \; m \; l$$

e e

$$M_l' := \int^{+} l \, d :=$$

e e o en of e f nc on e on fo o p y on n o e n fo nd n Le n z e n nd := e o n f $M \geq$ en

$$| \; | \; | \leq C \quad | \; |$$

e e nd ence e n e n e y con e en on fo o f o Le of e e o n

$$| \; | \leq C \quad | \; |^{M+\log_2 B}$$

e e

$$B := \sup_{\mathbf{R}} | \; e^{i} \; |$$

D e o e cond on e e o $B := M - \; -$ o e e e ence of o on of e e of eq on e nd fo o f o e e ence of e n e n nce e n f nc on co p c ppo e e e on y 7 7 d ee e e 7 d o e

∼

∝∞∈{∈{∞⟩∞∕∞∈{∞∈∞∞|∞|∈

e e

$$r \;\; = \sum_{l} r_l e^{il}$$

$$r_{\mathrm{even}} \;\; = \sum_{l} r \;\, {}_l e^{il} \qquad 7$$

nd

$$r_{\mathrm{odd}} \;\; = \sum_{l} r \;\, {}_{l+} \; e^{i \;\; l+ \;) \; =}$$

No c n

$$r_{\mathrm{even}} \;\; = r \qquad r$$

nd

$$r_{\mathrm{odd}} \;\; = r \qquad - r \qquad \text{e}$$

nd n e o n f o

$$r \;\; = r \qquad r \qquad \Big| - \quad r \qquad - r \qquad \text{e}$$

n y n e e

$$r \;\; = \qquad | \; r \qquad\qquad | \; r \qquad \text{e}$$

e n = n e e o n r = r nd e
n q ene of e o on of e nd fo o f o e n q ene of
e ep e en on of d d en e on r_l of e nd e con de e
ope o j de ned y e coe c en on e p ce $\mathbf{V}_j$ nd pp y o c en y
oo f nc on f nce r^j_l = $^j r_l$ e e e

$$_j f \;\; = \sum_{k \; Z} \;\; {}^j \sum_{l} r_l \, f_{j;k \;\; l} \qquad {}_{j;k} \qquad \text{e}$$

e e

$$f_{j;k \;\; l} = \;\; {}^{j=} \int_{}^{+} f \qquad {}^{j} \; - \qquad d \qquad \text{ee}$$

e n ee

$$f_{j;k \;\; l} =$$

d 7 7

e e — j nd | − | ≤ j n e n e nd n e
nd e o n

$$j f = \sum_{k \in \mathbb{Z}} \int_{+} f' \quad j;k \quad d \quad j;k$$

$$j \sum_{k \in \mathbb{Z}} \sum_{l} - r_l \int_{+} f'' \quad j;k \quad d \quad j;k \;!$$

c e → −∞ ope o j nd d d co nc de on oo f nc on n
nd d en e y o p o e = d d nd ence e o on o e
nd n q e e e on fo o no f o

Remark 2 e no e e p e on nd fo l nd l l = − l y
e p ed y c n n e o de of on n nd nd n od c n
e co e on coe c en $\sum_{i}^{L}$ n i i+n $\sum_{i}^{L}$ n i i+n nd $\sum_{i}^{L}$ n i i+n e
e p e on fo l e pec y p e l = e r l − r l

Examples. o e e p e e D ec e e e con c ed n
e co p e e coe c en k = M e e M e n e of n n
o en nd $L = M$ n e on e of

$$| = - \frac{M -}{M -} \int^{\mathbb{Z}} n^{M} \, d$$

e nd y co p n R n M d

$$| = -C_M \sum_{m}^{M} \frac{- \; m \; \text{co} \; -}{M \quad M \; - \quad -}$$

e e

$$C_M = \left[\frac{M -}{M - \quad e^{M}} \right]$$

y co p n e nd e e

$$m = \frac{- \; m \; C_M}{M \quad M \; - \quad -} \qquad \text{e e} = M$$

e no e y e of en o on of ne y e on co
e c en m n e on y con c on e coe c en r_l e on
n e e coe c en r_l e e e fo D ec e e e ed n e
of n n o en M e e e e e e e e fo en

j e d e d en

j ene e
of e p d o yno e

o n eq on of opo on e p e en e e fo D ec e e e
M — e

1 M —

— —— —— —

nd

r —— — r ——

e coe c en — — of e p e c n e fo nd n ny oo on n e c n y c o ce of coe c en fo n e c d en on

---

2 M —

— — — — —

nd

r — — r — r — — r_4 — —

---

3 M — e

— — — — — — —

nd

r — — r — r — —

r_4 — r — r — —

---

4 M —

— — — — — — — —

nd

r — — r — r — —

r_4 — r — —

nd

r_4

Coe c en fo M nd M c n e co p ed e co e pond n o p f o e fo o n e e o

Iterative algorithm for computing the coe cients r_l.

A y of o n eq on e nd e y o e n e e o e r nd r nd e e n e o eco p e r_l e y o e fy n e nd e ed d e o e c o ce of n z on e fo o n fo D ec e e e M co p ed n o d p y e coe c en $\{r_l\}$ e no e r $-r_l$ nd r

V.2 The operators d^n/dx^n in the wavelet bases

o e ope o d d e non nd d fo of e ope o d^n d^n co p e e y de e ned y ep e en on on e p ce **V** e y e coe c en

$$r_l^{(n)} = \int \frac{d^n}{d^n} d \quad \in \mathbf{Z}$$

o e n e y

$$r_l^{(n)} = \int | | e^{il} d$$

f e n e n o e ee e e o

		Coe cients
	i	c_i
M = 5	1	-0.82590601185015
	2	0.22882018706694
	3	-5.3352571932672E-

		Coe cients
	i	c_i
M = 8	1	-0.88344604609097
	2	0.30325935147672

Proposition V.2 **1. If the integrals in (5.52) or (5.53) exist, then the coe cients $r_l^{(n)}$, $l \in \mathbb{Z}$ satisfy the following system of linear algebraic equations**

$$r_l^{(n)} = \ ^{n} \left[r_{l} - \sum_{k}^{L} \ _{k} \left(r_{l} \ _{k+} \ r_{l+k} \right) \right]$$

and

$$\sum_{l} \ ^{n} r_l^{(n)} = - \ ^{n} n$$

where $_k$ **are given in (5.19).**

2. Let $M \geq n$, where M is the number of vanishing moments in (2.16). If the integrals in (5.52) or (5.53) exist, then the equations (5.54) and (5.55) have a unique solution with a nite number of non-zero coe cients $r_l^{(n)}$, namely, $r_l^{(n)} \neq$ for $-L \leq l \leq L -$. Also, for even n

$$r_l^{(n)} = r_{-l}^{(n)}$$

$$\sum_{l} \ ^{n} r_l^{(n)} = \quad n = \quad n - \qquad (5.57)$$

and

$$\sum_{l} r_l^{(n)} =$$

and for odd n

$$r_l^{(n)} = -r_{-l}^{(n)}$$

$$\sum_{l} \ ^{n} r_l^{(n)} = \quad n = \qquad \leq l \leq L$$

$A \quad e \; M$

e no e on e e e L = e e e e o n n o en M = do no e e o de e ponen ee e ep e en on of e d de e e on y f e n e of n n o en M =

e eq on fo co p n e coe c en $r_l^{n)}$ y e e ed n e en e p o e Le de e e eq on co e pond n o e fo d^n d^n d ec y f o e e e

$$r_l^{n)} = \int \sum_{k \in Z} | \quad | ^{n} \quad ^{n} e^{il} \, d$$

e efo e

$$r = \sum_{k \in Z} | \quad | ^{n} \quad ^{n}$$

e e

$$r = \sum_{l} r_l^{n)} e^{il}$$

n e e on

e

n o e nd de of nd n o e e en nd odd nd ce n ep e y e e

$$r = ^{n} \quad | r \quad | r$$

Le con de e ope o M on pe od c f nc on d f n f d

$$M f = \quad | f \quad | \quad |$$

n ee e e n e c e dence n f c of n e e nce ep
e en one of e d n e of co p n n e ee e
e e c e

N	[illegible]	[illegible]p
64	0.14545E+04	0.10792E+02
128	0.58181E+04	0.11511E+02
256	0.23272E+05	0.12091E+02
512	0.93089E+05	

e con o on ope o n e e e

n ec on e conde e co p on of e non nd d fo of con o on ope o . o con o on ope e q d e fo fo ep e en n e e ne on **V** e of e p e fo d e

nd e den y fo o f o
o d

nce e o en of e f nc on n eq on e e d o
po n q d e fo fo co p n e ep e en on of con o on
e ne e fo co p c y ppo ed e e fo o
e e nne fo e pec c o ce of e e e de c eqn
e e e f ed o en of e f nc on n e ef p pe fo
e de

e e e n od ce d en pp o c c con n of
ne e c eq on ec o y p o c cond on pe
c y p e f e y o of e ope o o o eneo of e
c e ope o co p e e y de ned y ep e en on on **V**

Le con de o e p e of c ope o e nd
ope o of f c on d en on o n d en on

VI.1 The Hilbert Transform

e pp y o e od o e co p on of e non nd d fo n
fo

$$\mathcal{H}f \quad y \quad \text{p} \quad \int \frac{f}{} \, d$$

e e p deno e p nc p e

e ep e en on of $\mathcal{H}$ on **V** de ned y e coe c en

$$r_{\mathbf{l}} \quad \int \mathcal{H} \quad d \quad \in \mathbf{Z}$$

c n n co p e e y de ne o e coe c en of e non nd d

$\mathcal{H} = \{A_{\mathbf{j}} \; B_{\mathbf{j}} \; \mathbf{j}\}_{\mathbf{j} \; \mathbf{Z}} \quad A_{\mathbf{j}} = A \quad B_{\mathbf{j}} = B$ nd $\mathbf{j}$ e e e

	l	Coe cients r_l	l	Coe cients r_l
M = 6	1	-0.588303698	9	-0.035367761
	2	-0.077576414	10	-0.031830988
	3	-0.128743695	11	-0.028937262
	4	-0.075063628	12	-0.026525823
	5	-0.064168018	13	-0.024485376
	6	-0.053041366	14	-0.022736420
	7	-0.045470650	15	-0.021220659
	8	-0.039788641	16	-0.019894368

e e e coe c en r_l of e n fo fo D ec e e e n n o en

e coe c en $r_l \in \mathbf{Z}$ n fy e fo o n y e of ne e c eq on

$$r_l = r_l - \sum_{k} (r_{l-k} + r_{l+k})$$

e e e coe c en k e en n n e nd e o n e y p o c of r_l fo e

$$r_l = O\left(\frac{1}{l^M}\right)$$

By e n n e of

$$r_l = \int |\ | \ n \ d$$

e o n $r_l = r_{-l}$ nd r e no e e coe c en r c nno e de e ned f o eq on e nd

o n e e y p o c cond on e co p e e coe c en r_l ny p e ed cc cy

Example.

e co p e e coe c en r_l of e n fo fo D ec e e e n n o en cc cy e coe c en fo e o ned n y p o c e no e $r_{-l} = r_l$ nd r

VI.2 The fractional derivatives

e e fo o n de n on of f c on de e

$$ {}_x f \;=\; \int^{+} \frac{- y_{+}}{} f\, y\, dy \qquad 7 $$

e e e con de $\neq$ f en 7 de ne f c on n de e
e ep e en on of $_x$ on **V** de e ned y e coe c en

$$ r_l \;=\; \int^{+} - \quad {}_x \quad d \quad \in \mathbf{Z} $$

p o ded n e e
e non nd d fo $_x = \{A_j\ B_j\ \ _j\}_{j\ Z}$ co p ed $A_j = {}^j A$ $B_j = {}^j B$ nd $_j = {}^j$ e e e e en $_{i\ l}$ $_{i\ l}$ nd $_{i\ l}$ of A B nd
e o ned f o e coe c en r_l

$$ {}_i = \sum_{k}^{L} \sum_{k'}^{L} \; {}_k \; {}_{k'} \; r_{i+k\ k'} $$

$$ {}_i = \sum_{k}^{L} \sum_{k'}^{L} \; {}_k \; {}_{k'} \; r_{i+k\ k'} $$

nd

$$ {}_i = \sum_{k}^{L} \sum_{k'}^{L} \; {}_k \; {}_{k'} \; r_{i+k\ k'} $$

e y o e fy e coe c en r_l fy e fo o n y e of ne
e c eq on

$$ r_l = \left[r_l - \sum_{k}^{L} {}_k \; r_{l\ k+} \; r_{l+\ k} \right] $$

e e e coe c en $_k$ e en n n e nd 7 e o n
e y p o c of r_l fo e

$$ r_l \;=\; \frac{}{} \frac{}{+} \; O \frac{}{+\ +\ M} \quad \text{fo} $$

$$ r_l \;=\; \quad \text{fo} \qquad e $$

Example.

	l	Coefficients	l	Coefficients
M = 6	-7	-2.82831017E-06	4	-2.77955293E-02
	-6	-1.68623867E-06	5	-2.61324170E-02
	-5	4.45847796E-04	6	-1.91718816E-02
	-4	-4.34633415E-03	7	-1.52272841E-02
	-3	2.28821728E-02	8	-1.24667403E-02
	-2	-8.49883759E-02	9	-1.04479500E-02
	-1	0.27799963	10	-8.92061945E-03
	0	0.84681966	11	-7.73225246E-03
	1	-0.69847577	12	-6.78614593E-03
	2	2.36400139E-02	13	-6.01838599E-03
	3	-8.97463780E-02	14	-5.38521459E-03

M p c on of ope o n e e e

VII.1 Multiplication of matrices in the standard form

e p c on of ce of C de on Zy nd nd p e do d e en ope o n e nd d fo eq e o O N o N ope on n dd on po e o con o e d of e n e nd y e n o ze o e en e n e p od c e o e o d of nd e efo e

$$|| - || \leq \quad 7$$

e nd de of 7 do n ed y o e p e f e co p e [4] en e o e one n c n d

VII.2 Multiplication of matrices in the non-standard form

e no o ne n o fo e p c on of e ope o n e non nd d fo ne o e e n e y deco p e e e n e p oce of p c on Le nd e o ope o

$$\mathbf{L} \quad \mathbf{R} \rightarrow \mathbf{L} \quad \mathbf{R} \qquad 7\,7$$

en e non nd d fo of nd $\{A_{\mathbf{j}} \; B_{\mathbf{j}} \; _{\mathbf{j}}\}_{\mathbf{j} \; \mathbf{z}}$ nd $\{A_{\mathbf{j}} \; B_{\mathbf{j}} \; _{\mathbf{j}}\}_{\mathbf{j} \; \mathbf{z}}$ e co p e e non nd d fo $\{A_{\mathbf{j}} \; B_{\mathbf{j}} \; _{\mathbf{j}}\}_{\mathbf{j} \; \mathbf{z}}$ of $-$

e ec e ope o of

n y e e e 7 of o e

7

e e

$$\sum_{j}^{n} A_j A_j \quad B_j \quad B_j \quad j \quad A_j B_j \quad j \quad A_j$$ 7

nd

$$n \quad n \quad \sum_{j}^{n} P_j \, B_j P_j$$ 7

e ope o n e 7 e c n on fo o n e p ce

$$A_j A_j \quad B_j \quad \mathbf{W}_j \to \mathbf{W}_j$$ 7 e

$$B_j \quad j \quad A_j B_j \quad \mathbf{V}_j \to \mathbf{W}_j$$ 7

$$j \quad A_j \quad \mathbf{W}_j \to \mathbf{V}_j$$ 7

nd e ope o n e 7

$$B_j \quad \mathbf{V}_j \to \mathbf{V}_j$$ 7 7

e e n 7 d d e f d 7 7 f

of ope on ed e c e e o o e p e e e o n e of
ope on ep p opo on o N
e e n ope o A_j B_j j n tot49 -410.315ac5 0 Td (36su8712.7097 0552

e e o n e e e

ef p c on o of ec on e econd fe o n e of e e o

VIII.1 An iterative algorithm for computing the generalized inverse

n o de o

p oced e nd e e on n e ee n ee n fo e co p on e e pe fo ed on n p c o on nd e ed o ne f o L N AC fo co p n e n e deco po on o e e ed e fo o n f n

$$A_{ij} = \begin{cases} \overline{i\ j} \\ \\ \end{cases}$$

e e N e cc cy e od e o 4 e en e of X_k e o 4 e e e c y e o ed f e e c e on

Size $N \times N$	SVD	FWT Generalized Inverse	L_2-Error
128 $\times$ 128	20.27 sec.	25.89 sec.	$3\ 1 \cdot 10^{-4}$
256 $\times$ 256	144.43 sec.	77.98 sec.	$3\ 42 \cdot 10^{-4}$
512 $\times$ 512	1,155 sec. (est.)	242.84 sec.	$6\ 0 \cdot 10^{-4}$
1024 $\times$ 1024	9,244 sec. (est.)	657.09 sec.	$7\ 7 \cdot 10^{-4}$
...	...	...	...
$2^{15} \times 2^{15}$	9.6 years (est.)	1 day (est.)	

Le de e e e e e o nd c n n e c f nc on c c ope o c n e p e en ed e c en y e fo p e do d en op e o N e c e nd e e pe fo nce of e o e epo ed ep e y

VIII.2 An iterative algorithm for computing the projection operator on the null space.

Le con de e fo o n e on

$$X_{k+} = X_k - X_k$$

$$X = A\ A$$

e e A e d o n nd c o en

en $-X_k$ con e e o P_{null} c n e o n e e d ec y o y co n n n n n ep e en on fo $P_{null} := -A\ AA^{\dagger}A$ e e on o co p e e ene zed n e $AA^{\dagger}$ e f p c on o e e e on e f fo de c of ope o e e co p e y e o fo e ene zed n e e po n d ence o e e e doe no eq e co p e y of e n e ope o on y of e po e of e ope o

VIII.3 An iterative algorithm for computing a square root of an operator.

Le de e n e on o con c o A nd A e e A fo p c y e f d o n nd non ne e de n e ope o e con de e fo o n e on

$$Y_{k+} := Y_k - Y_k X_k Y_k$$

$$X_{k+} := -X_k\quad Y_k A$$

$$Y := -A$$

$$X := -PA$$

7

e e c o en o e e e en e of $-A$ e n $\sqrt{\ }$ e eq ence X_k con e e o A nd Y_k o A By n $A := V\ DV$ e e D d on nd V n y e y o e fy o X_k nd Y_k c n e en $X_k := V\ P_k V$ nd $Y_k := V\ \ _kV$ e e P_k nd $_k$ e d on nd

$$_{k+} := \ _k - \ _k P_k\ \ _k$$

P e e

r n f f f d e y A f d f d

VIII.4 Fast algorithms for computing the exponential, sine and cosine of a matrix

e e ponen of o n ope o e ne nd co ne f nc on e on e o e con de ed n ny c c of ope o A n e c e of e ene zed n e e

X Co p n $F(u)$ n e e e e

n ec on e de e f d p e o fo co p n e e n n n e y d en e f nc on nd ep e en ed n e e An po n e p e O n y c e ene ze eo e of M Bony on e p op on of n e of o on of non ne eq on O n e c pp o c o e e no e e e pec de n e of pp c on of o

IX.1 The algorithm for evaluating u²

e n o o co p e Le p o ec $\in$ **L R** on p ce **V**$_j$ $\in$ **Z** o

$$_j = P_j \quad _j \in \mathbf{V}_j$$

n o de o deco p e e e e e e op c e e

$$- \;_n = \sum^{j}_{j} h \; P_j \; - P_j \; i = \sum^{j}_{j} P_j \; P_j \; P_j \; - P_j$$

n $P_j = P_j$ $_j$ e o n

$$- \;_n = \sum^{j}_{j} P_j \; _j \; _j$$

o

$$= \sum^{j}_{j} P_j \; _j \; \sum^{j}_{j} \; _j \; _j \; _n \qquad e$$

n e e e no n e c on e een d en e nd

o e n e c p po e need fo o e n e n e of e o ce

Befo e p oceed n f e e con de n e p e of e n

e e e fo o n e p c e on

$$
\begin{aligned}
& {}^{j}_{k} \quad — \quad j= \quad {}^{j}_{k} \\
& {}^{j}_{k} \quad — \quad j= \quad {}^{j}_{k} \\
{}^{j}_{k} \quad & {}^{j}_{k} \quad — \quad j= \quad {}^{j}_{k}
\end{aligned}
\qquad 7
$$

A o e pod c on e e e e zeo

p nd n e p c y n o

$$
— \sum_{j}^{n} \sum_{k \in Z} d^{j}_{k} \; {}^{j}_{k} \qquad \sum_{k \in Z} {}^{n}_{k} \; {}^{n}_{k}
$$

nd n 7 e o n f o e

$$
— \sum_{j}^{n} j= \sum_{k \in Z} d^{j}_{k} \; {}^{j}_{k} \; {}^{j}_{k} \qquad \sum_{j}^{n} j= \sum_{k \in Z} d^{j}_{k} \; {}^{j}_{k} \qquad n= \sum_{k \in Z} {}^{n}_{k} \; {}^{n}_{k}
$$

On deno n

$$
\begin{aligned}
d^{j}_{k} \quad & — \quad j= + \; d^{j}_{k} \; {}^{j}_{k} \\
{}^{j}_{k} \quad & — \quad j= \; d^{j}_{k} \\
{}^{n}_{k} \quad & — \quad n= \; {}^{n}_{k}
\end{aligned}
$$

e e e

$$
— \sum_{j}^{n} \sum_{k \in Z} d^{j}_{k} \; {}^{j}_{k} \qquad \sum_{j}^{n} \sum_{k \in Z} {}^{j}_{k} \; {}^{j}_{k} \qquad \sum_{k \in Z} {}^{n}_{k} \; {}^{n}_{k}
$$

e no e **if the coe cient d^{j}_{k} is zero then there is no need to keep the corresponding average ${}^{j}_{k}$** n o e o d e need o eep e e on y ne e n e e e e e e e coe c en d^{j}_{k} o p od c ${}^{j}_{k} d^{j}_{k}$ e n c n fo en cc cy

of coe c en c need o e o ed y e ed ced f e y o e n fo e p e

$$M^{j;j'}_{WWW} \; ' \; - \; j'= \int + \; j \; j' \; \; j \; j'_{k \; k'} \; \; {}_{j-j'} k \; l \; \; d$$

$$M^{j;j'}_{WWW} \; ' \; - \; j'= M^{j \; j'}_{WWW} \; - \; ' \; j \; j' \; -$$

o e e e o n c n ed c on n e n e of coe c en con e q ence of e f c e coe c en n dec y e d nce r — — ′ e een e e

f e n e of n c n coe c en d_k^j p opo on o e n e of e of N e e n e of ope on eq ed o e e e pp n
 oc e nece y o o e on y o e e k' c co ne e n c n coe c en d_k^j o p od ce non ze o con on e efo e c en o o e on y o k' fo c e e e e coe c en d_k^j c $| - '| \leq$ nd e p od c $k' d_k^j$ o e e e od of cc cy e n e need o o e e e on y n e ne o ood of n e

e n e of ope on fo e p nd n of e econd e n n o e e e p opo on o e n e of n c n en e nd e e e co p e e y o fo

Remark. e o fo e on n e e e o o e e e p od c of o f nc on nce

IX.2 The algorithm for evaluating F(u)

Le e n n n e y d en e f nc on n o de o deco p e e e e n e e op c e e

$$- \quad n \quad \sum_{j}^{n} \quad P_j \quad - \quad P_j \quad , \qquad 7$$

p nd n e f nc on n e y o e e e po n y e P d 7 7 e de p

e no e e e no e e econd de e of n e e e no e en
de e n n nd con de n e e nde of e e e n
n e o e e o n e e of M Bony e e o e o e e e
y oo e n n Bony e nce e e nde f c o j n e d
of j e y eep o e e o e e e nde y oo
no c e po n f e e n d n e n co p n
o epe ed pp c on of e o fo c en o co p e o
f nc on o e e e e e e e n y c d n e n con de n n
p c y n eo e c c e z n e

efe ence

B A pe p e ep e en on of oo ne ope o D e Y e n e y

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e A Co en D ec e nd C .

. , M e e of f eq ency c nne deco po on of e nd e e ode ec n c epo e Co n n e of M e c c ence Ne Yo n e y

. , Y Meye Le c c en q e e onde e e e e o en q d e C MAD n e e D p ne

. , Y Meye nc pe d nce de e e e enne e e e d ope e n w6ti3T