# FAST AND ACCURATE PROPAGATION OF COHERENT LIGHT

#### RYAN D. LEWIS, GREGORY BEYLKIN, AND LUCAS MONZÓN

Abstract. We describe a fast algorithm to propagate, for any user-specified accuracy, a time-harmonic electromagnetic field between two parallel planes separated by a linear, isotropic, and homogeneous medium. The analytic formulation of this problem (circa 1897) requires the evaluation of the so-called Rayleigh-Sommerfeld integral. If the distance between the planes is small, this integral can be accurately evaluated in the Fourier domain; if the distance is very large, it can be accurately approximated by asymptotic methods. In the large intermediate region of practical interest



nu r a) asured, or produced by a computational procedure such as phase recovery such analytic expansions and yield only a limit accuracy. urth r o nt on the stopic in  $\S C$  of the online supplement.

h n or an a urat propa at on a) or th arss n ar as su h as o pu tational holography component design and antinna design  $A$ . part  $u \text{ar}$ , interesting application area is X-ray diffraction more comparation area is  $x \text{ar}$ , and  $y \text{ar}$ related techniques where on attempts to order and a comparate of  $\alpha$  microscopic sample. p) row asuremts other a nitude of the magnitude of its magnitude of its difference in the magnitude of its difference problems are usually solved by the ratio vector of the sthat in  $\mu$  a light propagation a) or the herefore, the accuracy of the propagation algorithm ultimately limits the accuracy of the reconstructed in a speed of a propagation algorithm is obvous) also of critical importance of applications employees in the state  $\mathbf{a}$  is the state method in the state methods. h numerical algorithms that we use are designed to yield and user-specified  $\ln m$ 

a ura his n u includes controlled a controlled accuracy in the rapid computation of integrals.  $h$  the sthat we power this purpose specifically the USFF and generally the USFFT and generally the USFFT and general  $h$ a) Gaussian qualratures or band-limited functions for an significantly improve functions (and significantly improve functions) can significantly improve functions of  $\alpha$  and  $\alpha$  is denoted functions. the pror and and accuracy of  $\mathbf{r}$  and  $\mathbf{r}$  and  $\mathbf{r}$  and  $\mathbf{r}$  is the standard methods for  $\mathbf{r}$  between  $\mathbf{r}$  and  $\mathbf{r}$ s §§A and B o the online supplement on  $\mathbb{R}$ .

 $h$  pap  $r$  s or an as  $o$ ,  $b$  ws  $h$  n ssar at  $h$  at  $a$ ,  $p$   $r$ ,  $h$  nar s arrvw  $n \S$  srbournwalgorth  $n \S$  then suss its region of va) t  $n \S$  In  $\S$  we provide severation in all x a ples then summarize our rsults n § B ntrounths nwalorth whope to stult a ura provemts in computational optical systems by essentially eliminating numerical errors.

#### 2. Preliminaries

**The Rayleigh-Sommerfeld Formula.**  $h$  b hav or o a t  $\int$  harmonic  $\begin{bmatrix} \text{or} & \text{or$  Gvn the boundary data  $\boldsymbol{u} \cdot \boldsymbol{x}_t$  of  $\boldsymbol{x}_t$  we rewrite  $\boldsymbol{u}_t$  as

$$
u\cdot x,z \qquad \quad f\cdot y \quad K_z \quad \, x-y \quad \, dy,
$$

whrth  $a \rightarrow \nightharpoonup$  is  $c \rightarrow c$  in  $K_z$  (r) is vnb

$$
K_{z} r
$$
  $\frac{e^{i2 z} \overline{1+(r/z)^2}}{iz} \frac{1}{\sqrt{r/z^2}} \frac{1}{z^2-r/z^{\frac{3}{2}}}, r \ge 0$ 

D not n the Four r transform of the boundary data as

$$
\mathbf{f} \cdot \mathbf{p} \qquad \mathbf{f} \cdot \mathbf{x} \cdot e^{-i2 \cdot \mathbf{x} \cdot \mathbf{p}} \, \mathbf{dx},
$$

w wrt. nth Four r o an as

$$
\mathbf{u} \cdot \mathbf{x}, \mathbf{z} \qquad \mathbf{f} \cdot \mathbf{p} \quad \mathbf{K}_{\mathbf{z}} \qquad \mathbf{p} \quad e^{\mathbf{i} 2 - \mathbf{x} \cdot \mathbf{p}} \mathbf{d} \mathbf{p},
$$

wh r the Fourier transform of the Rayleigh-Sommerfeld kernel (cf. et al. (cf. et al. et n sthrns vnb

$$
K_z
$$
  $e^{i2 z} \overline{1-z}$   $\ge$ .

ur oa) s to valuate (2.2) accurately in such a way that the computational cost o s not n r as with the stance  $z$  It is contributed the spatial kernel  $K_z$  r s a h h) os jator un ton o  $r$  wh n z s s a) and that the Fourier domain kernel  $K_z$  sahighly oscillator untono when z star For an ph s all interesting the standard choices of the intermediate region,  $K_z$  (r) and  $K_z$ ar both h h ighly oscillator and the r thur a) of putation of  $\mathbf u$  using ther or (2.2) or (2.4) improvided in  $\int_a^b$  we will show how to approximate with ontrolled error and then  $\sinh$  and a factor and algorithm to apply then rsulting approximate Green's summature of the mainly function to boundary data. Our algorithm mainly material  $\lim_{n\to\infty}$ a r ss s the propagation problem for  $r$  intermediate and large values of z. For small values of z, it is well known that the problem may be solved using Fourier the set of  $\mathbf{z}$  is  $\mathbf{w}$  we show that the problem methods is solved using Fourier methods is  $\mathbf{z}$ an or vr and values of z, the problem may be solved using asymptotic methods s §§A and B o the online supplement on  $\sinh$ 

**Remark**  $\overrightarrow{G}$   $\overrightarrow{G}$   $\overrightarrow{v}$  n the normal derivative of the boundary data

$$
\begin{array}{c|cc}\n\text{-} \text{u} & \text{x, z} & \text{g} & \text{x} \n\end{array}
$$

Lore  $a \rightarrow bs$  or  $u \cdot a$  or the  $u \cdot a$  reads reads reads the Neumann problem reads reads not  $u \cdot a$  or  $t \cdot a$ 

$$
u \cdot x, z = -\int_{\mathbb{R}^2} g \cdot y \frac{e^{i2} R}{R} dy, \quad R = z^2 \quad x - y^{2 \frac{1}{2}}, \quad z > 0.
$$

th nor o at ons our approach salso applied is a in to valuat n

**Slepian Functions.** All ph s all realistic fields we wentually decay in spa an at the same time, are essentially band-limited in the Fourier domain. An appropriate mathematical description of such the such fields fields and his such fields  $\mathbb R$ collaborators n  $\Box$  by onsignma space-limiting and band-limiting and band-li nt rajoprator and using its eigenfunctions to identify a class of  $\mu$  in the functions that have ontrolled concentration in both the space and the Fourier domains.  $\mathfrak{D}$  and  $\mathfrak{D}$ t a) showed that this integral operator commutes with the differential operator o (ass a) ath at a)ph s s s r b n th pro at sph ro a) wave untons both operators share the same eigenfunctions.

For our purposes, we use eigenfunctions with  $\text{onto } \mathcal{W}$  on  $\text{intract on } n$  a square n the spatial domain and band-limited to a second-limited to a disk in the Fourier domain. The results in the Fourier domain. The spatial contract is spatially spatially spatially spatially spatially spatially spatially sp

 $\sum_{n=1}^{\infty}$  The Unequally Spaced Fast Fourier Transform. We need to evaluate transform.  $\operatorname{tr-} \operatorname{su-} \operatorname{so-} \operatorname{th-} \operatorname{or}$ 

$$
\begin{array}{c}\text{M} \\ \text{m} \ \text{m} \ \text{m}^\prime \ \textbf{f} \ \ \textbf{y}_{\text{mm}^\prime} \ \ \textbf{e}^{i\textbf{x} \cdot \textbf{y}_{\text{mm}^\prime}} \\ \text{m}, \text{m}^\prime = 1\end{array}
$$

at output points  $\mathbf{x}_{nn'}$   $\mathbf{x}_{n'}$ ,  $\mathbf{x}_{n'}$  where  $n, n'$ , ..., N. The u is and beevaluated rapidly, for any user-specified accuracy , using the USFFT (see [11, 3, with o putational complexity  $O \mathbb{N}^2$ 

 $\ln n$   $\mathbf{z}$   $\,$  s  $\,$   $\,$  u  $\,$   $\ln$   $\,$   $\,$  r than the spatial extent of  $\,$  f  $\,$  y  $\,$   $\,$  t  $\,$  s  $\,$  o  $\,$  on to  $\,$  a  $\,$ th urther approximation  $e^{i\frac{\pi}{z}||y||^2}$  which when use n (i.e. i.e. s to the Fraunhorso t salled  $\lim_{n \to \infty}$  and  $\lim_{n \to \infty}$  fraunhorso t salled far-field

$$
\mathbf{u} \cdot \mathbf{x}, \mathbf{z} = \frac{\mathbf{e}^{\mathbf{i} \cdot \mathbf{z}} - \mathbf{e}^{\mathbf{i} \cdot \mathbf{x}} \mathbf{z}}{\mathbf{i} \mathbf{z}} \mathbf{f} \mathbf{x}
$$

s  $s$ ,  $\frac{1}{3}$ , here  $\frac{1}{3}$  is the output field relation which relates the output field  $\frac{1}{3}$ to the sealed Fourier transform of the input field, is specially common in antenna s n and X ray ray to on order a more accurate approximation in the area  $\rightarrow$  s vnb

$$
\mathbf{u} \cdot \mathbf{x}_1 \mathbf{z} = -\frac{i z e^{i2} \cdot \frac{z^2 + ||\mathbf{x}||^2}{z^2 + ||\mathbf{x}||^2}}{\mathbf{z}^2 \cdot \mathbf{x}^2} \mathbf{f} = \frac{\mathbf{x}}{z^2 + |z|^2} , \quad z \to 0
$$

which may be valuated via the USFF we further is use the Fraunhore rapproximation of  $E$ FF we further approximation  $\mathbb{R}$  is the Fraunhore approximation of  $\mathbb{R}$  is the Fraunhore approximation of  $\mathbb{R}$  is the Fra at on an derive density on  $$B \text{ o}$  the online supplement.

#### 3. A New Algorithm for Fast and Accurate Light Propagation

In this ston we describe a asta algorithm to compute or a  $x$  propagation stan z and any user-species are  $\mathbf{a}$  uracy  $\mathbf{b}$  the field  $\mathbf{x}, \mathbf{z}$  in a square output wn ow **W**  $-\frac{\mathsf{w}}{2}$ ,  $\frac{\mathsf{w}}{2}$ <sup>2</sup> assu that the boundary data **f** has a report of  $\left[ \begin{array}{cc} a & b & a \end{array} \right]$ p.a. with its space-limited and band-limited version, as set of  $\frac{3}{3}$ . Hence,  $\frac{3}{3}$  $f$  s band-limited with some band-limit c and concentrated in a square aperture A  $-\frac{a}{2}, \frac{a}{2}$  $2$  so that a or n to wn to o put

$$
\mathbf{u} \cdot \mathbf{x}, \mathbf{z} \qquad \mathbf{f} \cdot \mathbf{y} \quad \mathbf{K}_{\mathbf{z}} \qquad \mathbf{x} \quad \text{H}_{\text{encm}}
$$

phas that  $n$  the sr a uracy K is scaled by the propagation stan z s n th a n tu o th m \ a s \  $z^{-1}$  a \ bn the optial axis. Inspired by the Fresnel Approximation, we rewrite the kernel as

$$
K_z\cdot r=-\frac{e^{i2-z}e^{i\frac{\pi}{z}r^2}}{iz}A_z\cdot r\ ,
$$

 $\leq r$ 

$$
A_{z} r \longrightarrow \frac{1}{r/z^{2}} \frac{1}{z^{2}} r/z^{2^{\frac{3}{2}}} e^{i2 z (\frac{1+(r/z)^{2}-1-\frac{1}{2}(r/z)^{2}}{2})}.
$$

wh r c s the bandlich to the input unit on  $f$  sn the bandlich  $\mathfrak{c}'$  we a rt the integrals in the integral or a service accuracy quantity quantures from  $\alpha$ h or

L t  $y_{mm'}$  y<sub>m</sub>,  $y_{m'}$  A m, m', ..., M b th  $M \times M$  t nsor prout r o qua ratur no swith the corresponding quadrature weights  $m \cdot m'$  . We sell the set of  $m$ an  $N \times N$  grid output locations  $x_{nn'}$   $x_{n'}$ ,  $x_{n'}$   $w$   $n, n'$  , ..., N then applie the quantum rotation of the theory is to the network of the matter and  $w$  and  $w$  is  $n \times N$ then apply the quadrature from  $h$  or approximation to the output of  $\lambda$  at the desired locations as

$$
u_{nn'} = \frac{e^{i2 z}}{iz} = \frac{1}{2} w_{mn'm'} m_{nn'm'} m_{nn'm''} \mathbf{f} \cdot \mathbf{y}_{mm'} e^{i2 \frac{z x_{nn'} \cdot y_{mm'}}{2}}.
$$

In the  $\textbf{N}\times\textbf{N}\times\textbf{M}\times\textbf{M}$  our<br>th-order tensors  $\textbf{T}^{( \ )}$ 

**Lemma 4.** Let  $\begin{pmatrix} 1 \ 0 \end{pmatrix}$ , **U**<sub>nq</sub>, and  $V_{mq}^{(1)}$ , where  $\begin{pmatrix} 1 & 1 \ 0 & 1 \end{pmatrix}$ , ..., L,

Theorem 5. The error of computing the field u from  $\overline{\phantom{a}}$  using  $\overline{\phantom{a}}$  is bounded by

$$
|u_{1}\mathbf{x}_{nn^{\prime}}_{l}z-u_{nn^{\prime}}|\quad\frac{K\quad\text{o}\quad R\quad f_{1}}{z}.
$$

In the expression or  $u_{nn'}$  in (3.15) allows us to valuate the field rapidly. We first apply  $\mathbf{Q}^{(-)}_{\mathsf{mm}'\mathsf{r}}$  as a pr  $]$  a tor to the input samples  $\mathsf{f}$  (ymm'

o illustrat the difference between  $\mathbf{w}_{\text{max}}$  and  $\mathbf{z}_{\text{min}}$  or our the and  $\mathbf{w}'_{\text{max}}$  and  $z'_{\text{min}}$  or the Fr snel approximation, let us hoos  $z^{-3}$  I a wavelengths then at r propagating  $z \times 6$ 







Figure 5.5. Co parson oth a ntu oth  $\phi$  or a oal point  $\circ$  o the optical axis of puteent by our algorithm correct to  $ts \rightarrow t$  and by the Fresnel approximation (right) of enhance contrast  $w$  plot the square root of the magnitude magnitude,  $w$  and  $w$  $|{\bf u} \times_1, {\bf x}_2 |^{1/2}$ h Fr sn ) approximation shifts the location of the focal spot and blurs the boundaries between the manufold and side of  $\mathcal{I}$ a so the bottom-right plot in Figure



n Furto bttra) nth pasoth solid and dashed lines. In ortunately, our xample shows that the Fr snul approximation in orrectly computes the shape o the oal spot, naddom to its position of parthermults between the main  $\label{eq:1} \begin{array}{lll} \text{in} & \text{in} &$ 

Representative Examples of Computational Cost. In o putational ost o our a) or the p n s on the number of  $\mathfrak{F}\mathrm{FF}\,$ s required in **R**  $\mathbf{R}^{(1)}$   $\cdots$   $\mathbf{R}^{(L)}$  wh r **L** sthe number of terms needed to approximate the  $k = \min_{i=1}^n k_i$ . As turns out  $R$  decreases with increasing z, which is expected in  $R$  $\gamma$ n the application of the  $a \rightarrow \frac{h}{c}$  or  $\gamma$  all  $x$  ally reduces to

## **Conclusions**

have srb a ast alorth or the propagation of coherent light between para led planes separated by a linear sotropic and homogeneous medium. In contrast to urr nt a) or the sour a) or the a heves and user-specified accuracy. As a ons qun w an rap i and accurately compute the in non-paraxial regions, i.e., regions are regions for the optical axis, with computational complexity proportional to that of the FFT. The overall result is a fast algorithm that can achieve an user-specified accuracy over a large computational domain.

## Acknowledgments

than Dr. Bradley Alpert roth  $\mathbb{I}$  for providing many useful comments an su st ons

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