FAST AND ACCURATE PROPAGATION OF COHERENT LIGHT

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Abstract. **We describe a fast algorithm to propagate, for any user-specified accuracy, a time-harmonic electromagnetic field between two parallel planes separated by a linear, isotropic, and homogeneous medium. The analytic formulation of this problem (circa 1897) requires the evaluation of the so-called Rayleigh-Sommerfeld integral. If the distance between the planes is small, this integral can be accurately evaluated in the Fourier domain; if the distance is very large, it can be accurately approximated by asymptotic methods. In the large intermediate region of practical interest**

$$u(x,z) = -\frac{}{\pi} \int_{\mathbb{R}^2} f(y) \frac{\partial}{\partial z} \left(\frac{e^{i2\pi R}}{R} \right) dy, \quad z > \ ,$$

wh r R $\sqrt{z^2 \quad x - y^2}$ G v n th u y, f y n th p an z s r b s th u x, z $z >$ that sat s s th o r ra at on on t on Expr ss n a stan s n wav n th s w not that th propa at on nu r a. asur or pro u b a o putat ona pro ur su h as phas r ov r su h ana t xpans ons an on a t a ura urth r o nt on th s top n §C o th on n supp nt

h n or an a urat propa at on a or th ar s s n ar as su h as o pu tat ona ho o raph opt a o pon nt s n an ant nna s n A part u ar nt r st n app at on ar a s X ra ra t on ros op an r at t hn qu s wh r on att pts to or an a o a ros op sa p ro asur nts o th a n tu o ts ra t on patt rn h s nv rs prob s ar usua so v b t rat v tho s that n u a ht propa at on a or th h r or th a ura o th propa at on a or th u t at ts th a ura o th r onstru t a h sp o a propa at on a or th s obv ous a so o r t a portan or app at ons p o n t rat v tho s

h nu r a a or th s that w us ar s n to an us sp a ura h s n u s ontro a ura n th rap o putat on o nt ra s h tho s that w p o or th s purpos sp a th FF an n a Gauss an qua ratur s or ban t un t ons an s n ant prov th p r or an an a ura o v n th stan ar tho s or ht propa at on s §§A an B o th on n supp nt

h pap r s or an as o ows h n ssar ath at a pr nar s ar r v w n § s r b our n w a or th n § th n s uss ts r on o va t n § In § w prov s v ra nu r a xa p s th n su ar our r su ts n § B ntro u n th s n w a or th w hop to st u at a ura prov nts n o putat ona opt a s st s b ss nt a nat n nu r a rrors

Preliminaries

The Rayleigh-Sommerfeld Formula. h b hav or o a t har on tro a n t n a n ar sot s o a r r F

G v n th boun ar ata **u** x, **f** x w r wr t as

$$u\ x, z\quad f\ y\ K_z\ x - y\ dy,$$
$\mathbb{R}^2$

wh r th a r rn K_z **r** s v n b

$$K_z\ r\quad \frac{e^{i2\pi z\ \overline{1+(r/z)^2}}}{iz}\ \ \frac{}{r/z\ ^2}\quad \frac{i}{\pi z\quad r/z\ ^2\ \frac{3}{2}},\quad r \geq\ .$$

D not n th Four r trans or o th boun ar ata as

$$\hat{f}\ p\quad f\ x\ e^{-i2\pi x \cdot p}\, dx,$$
$\mathbb{R}^2$

w wr t n th Four r o a n as

$$u\ x, z\quad \hat{f}\ p\ \hat{K}_z\ p\ e^{i2\pi x \cdot p}\, dp,$$
$\mathbb{R}^2$

wh r th Four r trans or o th a r rn an r n s th r n s v n b

$$\hat{K}_z\ \rho\quad e^{i2\pi z\ \overline{1-\rho^2}},\quad \rho \geq\ .$$

ur oa s to va uat a urat n su h a wa that th o putat ona ost o s not n r as w th th stan **z** It s ar that th spat a rn K_z **r** s a h h os ator un t on o **r** wh n **z** s s a an that th Four r o a n rn $\hat{K}_z$ **ρ** s a h h os ator un t on o **ρ** wh n **z** s ar For an ph s a nt r st n ho s o th stan **z** n th nt r at r on K_z **r** an $\hat{K}_z$ **ρ** ar both h h os ator a n th r t nu r a o putat on o **u** us n th r or pra t a In § w w show how to approx at w th ontro rror an th n s r b a ast an a urat a or th to app th r su t n approx at Gr n s un t on to boun ar ata ur a or th a n a r ss s th propa at on prob or nt r at an ar va u s o **z** For s a va u s o **z** t s w nown that th prob a b so v us n Four r tho s an or v r ar va u s o **z** th prob a b so v us n as ptot tho s s §§A an B o th on n supp nt

Remark . G v n th nor a r vat v o th boun ar ata

$$\frac{\partial}{\partial z} u\ x, z\ \Big|_{z=0}\quad g\ x\ ,$$

Lor a h s or u a or th u ann prob r a s

$$u\ x, z\quad -\frac{}{\pi}\int_{\mathbb{R}^2} g\ y\ \frac{e^{i2\pi R}}{R}\, dy,\quad R\quad z^2\quad x - y\ ^2\ \frac{1}{2},\ z >\ .$$

th nor o at ons our approa h s a so app ab to va uat n

Slepian Functions. A ph s a r a st s ust v ntua a n spa an at th sa t ar ss nt a ban t n th Four r o a n An appropr at ath at a s r pt on o su h s was n t at b p an an h s o aborators n b ons r n a spa t n an ban t n nt ra op rator an us n ts n un t ons to nt a ass o un t ons that hav ontro on ntrat on n both th spa an th Four r o a ns p an t a show that th s nt ra op rator o ut s w th th r nt a op rator o ass a ath at a ph s s s r b n th pro at sph ro a wav un t ons both op rators shar th sa n un t ons

For our purpos s w us n un t ons w th ontro on ntrat on n a squar n th spat a o a n an ban t to a s n th Four r o a n h

The Unequally Spaced Fast Fourier Transform. n to va uat tr ond tr su s o th or

$$\sum_{m,m'=1}^{M} \tau_m \tau_{m'} f \; y_{mm'} \; e^{i x \cdot y_{mm'}}$$

at output points $x_{nn'}$ $x_n, x_{n'}$ wh r n, n' $, \dots, N$ u h su s an b va uat rap or an us sp a ura us n th FF s w th o putat ona o p x t $\mathcal{O}$ N^2

h n **z** s u h ar r than th spat a xt nt o **f** **y** t s o on to a th urth r approx at on $\mathbf{e}^{\mathbf{i}\frac{\pi}{z}\|\mathbf{y}\|^2}$ wh h wh n us n a s to th Fraunho r so t s a ar approx at on

$$\mathbf{u}\ \mathbf{x},\mathbf{z} \quad \frac{\mathbf{e}^{\mathbf{i}2\pi\mathbf{z}}\mathbf{e}^{\mathbf{i}\frac{\pi}{z}\|\mathbf{x}\|^2}}{\mathbf{iz}}\hat{\mathbf{f}}\left(\frac{\mathbf{x}}{\mathbf{z}}\right)$$

s § h Fraunho r approx at on wh h r at s th output to th s a Four r trans or o th nput s sp a o on n ant nna s n an X ra ra t on ros op A or a urat approx at on n th ar s v n b

$$\mathbf{u}\ \mathbf{x},\mathbf{z} \quad -\frac{\mathbf{ize}^{\mathbf{i}2\pi\sqrt{\mathbf{z}^2+\|\mathbf{x}\|^2}}}{\mathbf{z}^2 \quad \mathbf{x}^{\ 2}}\hat{\mathbf{f}}\left(\frac{\mathbf{x}}{\mathbf{z}^2 \quad \mathbf{x}^{\ 2}}\right), \quad \mathbf{z} \to \quad ,$$

wh h a b va uat v a th FF urth r s uss th Fraunho r approx at on an r v n §B o th on n supp nt

A New Algorithm for Fast and Accurate Light Propagation

In th s s t on w s r b a ast a or th to o put or a x propa at on stan **z** an an us sp a ura **>** th **u** **x,z** n a squar output w n ow $\mathbf{W} \quad \left[-\frac{\mathbf{w}}{2},\frac{\mathbf{w}}{2}\right]^2$ assu that th boun ar ata **f** has a r a b n r p a w th ts spa t an ban t v rs on as s r b n § H n **f** s ban t w th so ban t **c** an on ntrat n a squar ap rtur $\mathbf{A} \quad \left[-\frac{\mathbf{a}}{2},\frac{\mathbf{a}}{2}\right]^2$ so that a or n to w n to o put

$$\mathbf{u}\ \mathbf{x},\mathbf{z} \quad \int_{\mathbf{A}} \mathbf{f}\ \mathbf{y}\ \mathbf{K}_{\mathbf{z}}\ \mathbf{x} - \mathbf{K}$$

phas that n th s r a ura K s s a b th propa at on stan z s n th a n tu o th rn a s z^{-1} a on th opt a ax s Insp r b th Fr sn approx at on w r wr t th rn as

$$K_z \; r \qquad \frac{e^{i2\pi z} e^{i\frac{\pi}{z} r^2}}{iz} A_z \; r \; ,$$

wh r

$$A_z \; r \qquad \frac{}{r/z^{\;2}} \qquad \frac{i}{\pi z \qquad r/z^{\;2}\,\frac{3}{2}} \; e^{i2\pi z\left(\overline{1+(r/z)^2}-1-\frac{1}{2}(r/z)^2\right)}.$$

wh r $\mathbf{c}$ s th ban t o th nput un t on $\mathbf{f}$ s n th ban t $\mathbf{c}'$ w r t th nt ra s n or a s r a ura $_{\mathbf{Q}}$ us n th qua ratur s ro h or

L t $\mathbf{y}_{\mathbf{mm}'}$ $\mathbf{y}_{\mathbf{m}},\mathbf{y}_{\mathbf{m}'}$ $\mathbf{A}$ $\mathbf{m},\mathbf{m}'$ $,\ldots,\mathbf{M}$ b th $\mathbf{M}\times\mathbf{M}$ t nsor pro u t r o qua ratur no s w th th orr spon n qua ratur w hts $\boldsymbol{\tau}_{\mathbf{m}}\boldsymbol{\tau}_{\mathbf{m}'}$ s t an $\mathbf{N}\times\mathbf{N}$ r o output o at ons $\mathbf{x}_{\mathbf{nn}'}$ $\mathbf{x}_{\mathbf{n}},\mathbf{x}_{\mathbf{n}'}$ $\mathbf{W}$ $\mathbf{n},\mathbf{n}'$ $,\ldots,\mathbf{N}$ th n app th qua ratur ro h or to th nt ra s n an obta n an approx at on to th output at th s r o at ons as

$$\mathbf{u}_{\mathbf{nn}'} \quad \frac{\mathbf{e}^{\mathbf{i}2\pi\mathbf{z}}}{\mathbf{iz}} \sum_{\ell=1}^{\mathbf{L}} \mathbf{w}_{\ell} \sum_{\mathbf{m},\mathbf{m}'=1}^{\mathbf{M}} \boldsymbol{\tau}_{\mathbf{m}}\boldsymbol{\tau}_{\mathbf{m}'} \mathbf{T}^{(\ell)}_{\mathbf{nn}'\mathbf{mm}'} \mathbf{f} \; \mathbf{y}_{\mathbf{mm}'} \; \mathbf{e}^{\mathbf{i}2\boldsymbol{\beta}_{\ell}\mathbf{x}_{nn'}\cdot\mathbf{y}_{mm'}}.$$

In th $\mathbf{N}\times\mathbf{N}\times\mathbf{M}\times\mathbf{M}$ ourth or r t nsors $\mathbf{T}^{(\ell)}$

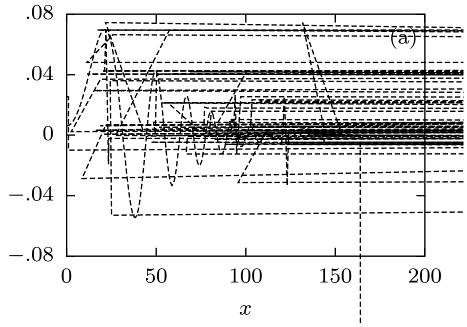
Lemma 4. Let $\sigma_q^{(\ell)}$, $U_{nq}^{(\ell)}$, and $V_{mq}^{(\ell)}$, where ℓ ˇ$, \ldots, L$,

Theorem 5. **The error of computing the field u from using is bounded by**

$$|u\,(x_{nn'}, z) - \bar{u}_{nn'}| \quad \frac{\Box_K \quad \Box_Q \quad \Box_R \quad f \quad _1}{z}.$$

h xpr ss on or $\bar{u}_{nn'}$ n allows us to valuat th rap rst app $Q^{(\Box)}_{mm'r}$ as a pr] a tor to th nput sa p s f $y_{mm'}$

o illustrat th r n b tw n $\mathbf{w}_{max}$ an $\mathbf{z}_{min}$ or our tho an $\mathbf{w}'_{max}$ an $\mathbf{z}'_{min}$ or th Fr sn l approx at on it us hoos $^{-3}$ I $\mathbf{a}$ wav in th s th n a t r propa at n $\mathbf{z}$ $\times$ ^{6}


(a)
.08
.04
0
−.04
−.08
0
50
100
150
200
x

20

0

-20

-40

-60

(b)

−200 −100 0 100 200

−

x

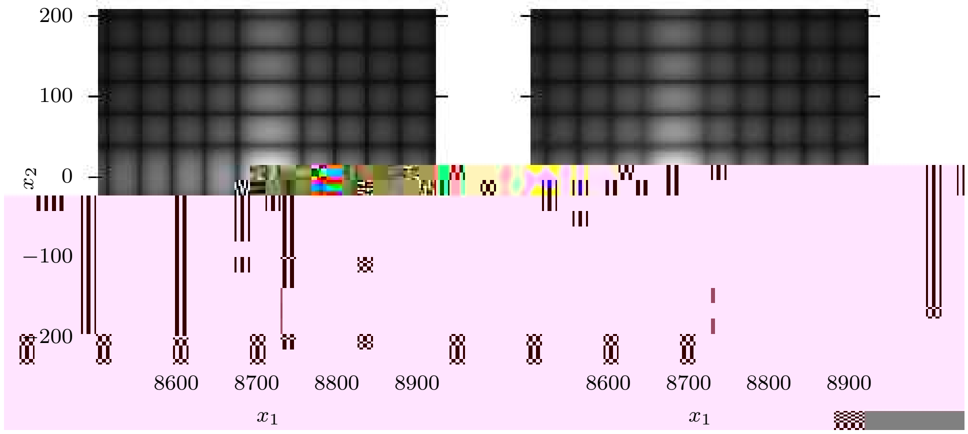
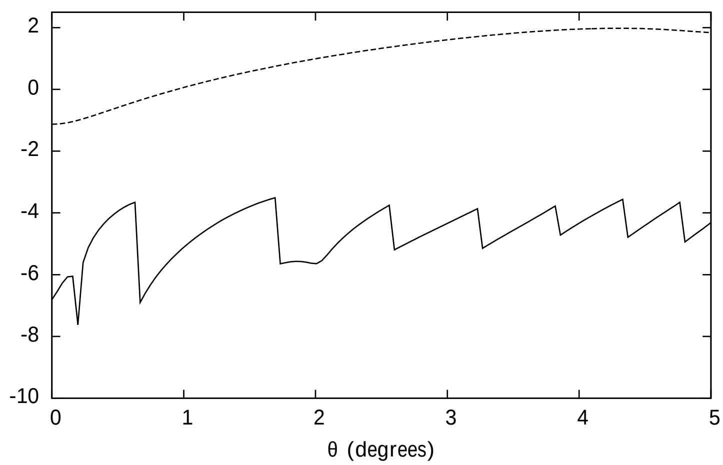


Figure 5.5. Co par son o th a n tu o th or a o a po nt ° o th opt a ax s o put b our a or th orr t to ts t an b th Fr sn approx at on r ht o nhan ontrast w p ot th squar root o th a n tu $|\mathbf{u}\ \mathbf{x}_1, \mathbf{x}_2|^{1/2}$ h Fr sn approx at on sh ts th p at on o th o a spot an b urs th boun ar s b tw n th a n b an s b s a so th botto r ht p ot n F ur



n F ur to b tt r a n th p a s o th so an ash n s n ortunat our xa p shows that th Fr sn approx at on n orr t o put s th shap o th o a spot n a t on to ts pos t on o par th nu s b tw n th a n b an s b s n th botto] r ht p ot n F ur

Representative Examples of Computational Cost. h o putat ona ost o our a or th p n s on th nu b r o FF s r qu r n $\mathbf{R}$ $\mathbf{R}^{(1)}$ ... $\mathbf{R}^{(L)}$ wh r $\mathbf{L}$ s th nu b r o t r s n to approx at th rn n As t turns out $\mathbf{R}$ r as s w th n r as n $\mathbf{z}$ wh h s xp t $\mathbf{z}$ n th app at on o th a o r rn as ptot a r u s to

Conclusions

hav s r b a ast a or th or th propa at on o oh r nt ht b tw n para p an s s parat b a n ar sotrop an ho o n ous u In on trast to urr nt a or th s our a or th a h v s an us sp a ura As a ons qu n w an rap an a urat o put th n non parax a r ons r ons ar ro th opt a ax s w th o putat ona o p x t pro port ona to that o th FF h ov ra r su t s a ast a or th that an a h v an us sp a ura ov r a ar o putat ona o a n

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