nodes lie on a polar grid, so that

$$\mathbf{p} =$$

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$\sqrt{z^2 + \|x\|^2} = z\left(1 + \frac{1}{2}\frac{\|x\|^2}{z^2} + \cdots\right)$ in the denominators and exponent of (B.4), respectively, yielding the standard Fraunhofer approximation (cf. [12, §4.3]),

$$u(x, z) \approx \frac{e^{2\pi i z} e^{\frac{i\pi}{z}\|x\|^2}}{iz} \hat{f}\left(\frac{x}{z}\right). \tag{B.5}$$

However, this additional approximation is not advisable since, unlike (B.4), the Fraunhofer approximation is only valid for points close to the the optical axis, i.e., $\|x\| \ll z$. There is no computational advantage to be gained by using (B.5), with its restricted region of validity, instead of (B.4), valid for all x provided that z is sufficiently large, since both of these formulae may be evaluated at the same cost with a single USFFT.

Appendix C. Supplementary Information and Statistics

C.1. Algorithm for Approximation by Exponential Sums. We approximate, for any user-specified accuracy ϵ, a smooth function $f(x)$, $0 \le x \le 1$, by a linear combination of exponentials,

$$\left| f(x) - \sum_{\ell=1}^{L} w_\ell e^{-\eta_\ell x} \right| \le \epsilon, \quad x \in [0, 1], \tag{C.1}$$

where the number of complex-valued weights w

Compute the roots $y_{\square}$ of the con-eigenpolynomial $u(z) = \sum_{n=0}^{N} u_n z^n$

The rightmost factor in (C.6) depends approximately linearly on B and only weakly on ϵ_K, so that $L = \mathcal{O}\left(y^4 \log \epsilon_K^{-1}\right)$. Since the number of terms grows rapidly with y, we require that $y \leq 2.62$ to ensure that the approximation is efficient. This implies that the maximum output window is given by (4.1), and we assume that a, w, and z satisfy $y \leq 2.62$ for the remainder of this section.

The same integral-based techniques that lead to (C.6) also yield the bounds

$$\alpha_\ell \leq \frac{B + D}{r_{max}^2} \tag{C.8}$$

and

$$\beta_\ell \leq \frac{\pi}{z} + \frac{D}{r_{max}^2}, \tag{C.9}$$

where

$$D = D(B, \epsilon_K) = \sqrt{B^2 - 8B\left(\log \epsilon_K + \log 2 - \pi B\right)}.$$

(Recall from §3(b) that $\alpha_\ell = -\mathcal{R}e\,\eta_\ell$ and $\beta_\ell = \mathcal{I}m\,\eta_\ell - \frac{\pi}{z}$, where η_ℓ, $\ell = 1, \ldots, L$, are the exponents used to approximate the Rayleigh-Sommerfeld kernel.) We will use these bounds below to determine the number of required input samples M^2 and the number of terms $R^{(\ell)}$ in the approximations (3.14). To simplify the computations

Because $|x'| \leq \sqrt{2}$, the bandlimit of the factor $e^{i\frac{aw}{2}x'\cdot y'}$ is $c_3 = \frac{\pi aw}{\sqrt{2}z}$, where we used the approximation $\beta \approx \frac{\pi}{z}$.

Thus, the bandlimit of the integrand is approximately

$$c' = c + c_2 + c_3 = c + \frac{a^2}{2\sqrt{2}z} + \frac{\pi aw}{\sqrt{2}z}. \tag{C.10}$$

From [7], we have that, for a desired accuracy ϵ_Q, the number of samples required to evaluate the integrals (3.8) satisfies $M^2 = \mathcal{O}\left((c')^2 \log^2 \epsilon_Q^{-1}\right)$.

C.3.3. *[illegible] to Appro[illegible] nsors.* Now let us use the bound (C.8) to estimate the number of terms required in the approximations (3.14). The tensors $S^{(\ell)}$ in (3.11) are discrete approximations of the functions

$$S^{(\ell)}(x,y) = e^{2\alpha_\ell xy}, \quad x \in \left[-\frac{w}{2}, \frac{w}{2}\right], y \in \left[-\frac{a}{2}, \frac{a}{2}\right], \text{ and } \ell = 1, \ldots, L,$$

which have the Chebyshev expansions

$$e^{2\alpha_\ell xy} = J_0(-i\alpha_\ell ax) + 2\sum_{n=1}^{\infty} i^n J_n(-i\alpha_\ell ax)\, T_n\left(\frac{2y}{a}\right),$$

where J_n is the n-th order Bessel function of the first kind and T_n is the n-th degree Chebyshev polynomial of the first kind. For fixed x, the magnitude of the Bessel functions decay super-exponentially as $n \to \infty$. In fact, using [1, (9.1.62)], we have the bound

$$|J_n(-i\alpha_\ell ax)| \leq \frac{|\alpha_\ell ax|^n e^{|\alpha_\ell ax|}}{2^n n!}.$$

Now observe that $|\alpha_\ell ax| \leq D\frac{aw}{(a+w)^2}$, where we used (C.8) and the fact that $r_{max} = (a + w)/\sqrt{2}$. For some desired accuracy ϵ_R, let P be the smallest integer such that $\left|J_n\left(-iD\frac{aw}{(a+w)^2}\right)\right|^2 \leq \epsilon_R$ for all $n \geq P$. Then we may estimate the number of terms $R^{(\ell)}$ in (3.14) as

$$R^{(\ell)} \leq (P+1)^2 = \mathcal{O}\left(\log \epsilon_R^{-1}\right).$$

If we assume that the output window is at least as large as the input aperture, i.e., $w \geq a$, then the argument of the Bessel function satisfies

$$\left|D\frac{aw}{(a+w)^2}\right| \leq \frac{D}{4},$$

and it is easy to verify that the numerical rank R of each matrix $S^{(\ell)}$ satisfi5 [(i)-4.7933(D)-3.19063]TJR24 experimentally-determined data). Such a scheme does not provide a mechanism to control error, i.e., the user cannot specify any desired accuracy $\epsilon > 0$ in advance and be assured that the level of error in the computed solution is bounded by ϵ. We note that improvements to the Fresnel approximation can be obtained for special types of boundary data—for example [5] describes such a scheme for the special case of a converging spherical wave. However, we are not aware of any numerical method based on the Fresnel approximation that can propagate arbitrary boundary data with controlled error.

Although we state that our method can be viewed as a generalization of the Fresnel approximation—indeed, it can be— it is, in fact, not directly related to this approximation (including the alternative form discussed in, e.g., [10]). Our results provide an algorithm to accurately compute the Rayleigh-Sommerfeld integral with any user-specified accuracy and it just happens that the form of approximation resembles the Fresnel approximation. We think it is worthwhile to point out this resemblance but, in essence, our paper does not deal with the Fresnel approximation as such.

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A a an D Z Arov an G Kr n *n t H n tr s n n r C r t o or F r n r pro s* Fun ona Ana r o n **2** no