Preliminary Exam
Partial Differential Equations
9:00 AM - 12:00 PM, Jan. 11, 2024
Newton Lab, ECCR 257

Student ID (do NOT write your name):

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There are five problems. **Solve four of the five problems.**
Each problem is worth 25 points.

A sheet of convenient formulae is provided.

3. **Wave Equation.** Consider the following initial-boundary value problem on the domain $D = \{(x,t) : t \in \mathbb{R}^+, x \in \mathbb{R}^+, x > t/\alpha\}$, where $\alpha > 1$:

$$
\begin{aligned}
u_{tt} &= u_{xx}, & x > t/\alpha,\ t > 0, \quad (3)\\
u(x,0) &= \varphi(x), & x > 0, \quad (4)\\
u_t(x,0) &= \psi(x), & x > 0, \quad (5)\\
u(x,\alpha x) &= f(x), & x > 0, \quad (6)
\end{aligned}
$$

with $\varphi, \psi, f \in C^2(\mathbb{R}_0^+)$.

(a) Find the solution $u(x,t)$.

(b) Find sufficient conditions on $\psi$, $\varphi$, and $f$ so that the solution is continuous in $D$.

4. **Laplace's Equation/Green's Functions.** Consider the Neumann problem on the disk in $\mathbb{R}^2$

$$
\begin{aligned}
&\Delta u(\mathbf{x}) = 0, \quad \mathbf{x} \in B(0,1) = \left\{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| < 1\right\},\\
&\frac{\partial u}{\partial r}(r = 1, \theta) = g(\theta), \quad \theta \in [0, 2\pi], \quad g(0) = g(2\pi), \quad g'(0) = g'(2\pi),
\end{aligned} \quad (7)
$$

where $r = |\mathbf{x}|$ and $\theta = \arctan(x_2/x_1)$ are polar coordinates and $g \in C^2(0, 2\pi)$.

(a) What is a necessary condition for the solution to exist? What additional condition can be applied to make the solution unique? Prove that under this condition, the solution is unique.

(b) Solve the Neumann problem in (7).

(c) Using your solution from (b), identify the Neumann function for the unit disk. Hint: $\sum_{n=1}^{\infty} R^n/n = -\log(1-R)$ for $|R| < 1$.

5. **Solution methods.** Let $\Omega = (0,1) \times \mathbb{R}^+$, and assume that $u(x,t) \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ satisfies

$$
\begin{aligned}
u_t &= u_{xx} + f(x)e^{-t}, & 0 < x < 1,\ t > 0, \quad (8)\\
u(x,0) &= 0, & 0 < x < 1, \quad (9)\\
u(0,t) &= u(1,t) = 0 & t > 0, \quad (10)
\end{aligned}
$$

where $f \in C^1([0,1])$.

(a) Use Duhamel's principle to find a formal solution to the initial boundary value problem in terms of $f_n$, the Fourier coefficients of $f(x)$.

(b) Prove that the solution is unique.

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