Preliminary Exam
Partial Differential Equations
9:00 AM - 12:00 PM, Jan. 11, 2024
Newton Lab, ECCR 257

Student ID (do NOT write your name):

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| # | possible | score |
|---|---|---|
| 1 | 25 | |
| 2 | 25 | |
| 3 | 25 | |
| 4 | 25 | |
| 5 | 25 | |
| Total | 100 | |

There are five problems. **Solve four of the five problems.**
Each problem is worth 25 points.

A sheet of convenient formulae is provided.

1. **Method of characteristics.** Consider the inviscid Burger's equation

$$u_t + uu_x = 0 \tag{1}$$

on the domain $\Omega = \mathbb{R} \times \mathbb{R}^+$ with initial conditions

$$u(x, 0) = u_0(x) = \begin{cases} 1, & x \le 0, \\ 1 - x, & 0 < x \le 1, \\ 0, & 1 < x. \end{cases} \tag{2}$$

(a) Find the time and position at which a shock forms.

**Solution:** The characteristic equations are

$$\frac{dt}{d\tau} = 1, \tag{3}$$
$$\frac{dx}{d\tau} = u, \tag{4}$$
$$\frac{du}{d\tau} = 0, \tag{5}$$
$$\tag{6}$$

which gives, using the initial data $(x, t, u) = (s, 0, u_0(s))$,

$$t = \tau, \tag{7}$$
$$x = ut + s, \tag{8}$$
$$u = u_0(s). \tag{9}$$

Thus, the solution $u$ satisfies the implicit equation $u = u_0(x - ut)$. To find the location of the shock, we differentiate with respect to $x$ and solve for $u_x$, finding

$$u_x = \frac{u_0}{1 + u_0 t}. \tag{10}$$

Thus, a characteristic emanating from the initial point uninitialead.21 0 05.992 0 T 47(orth) 0 a oi(u)]TJ/c290.061(w)27(orth)-32whe.95

we conclude that all characteristics emanating from $(0, 1)$ produce a shock at $t_s = 1$. The position of the shock for the characteristic starting at $x_0 = s \quad (-1, 1)$ can be found by setting $t = t_s = 1$ and $u = u_0(s) = 1 - s$ in Eq. (8), which gives $x_s = (1 - s)1 + s = 1$. Therefore the shock forms at $(x_s, t_s) = (1, 1)$.

(b) Find the subsequent trajectory of the discontinuous shock by applying the Rankine-Hugoniot condition

$$s(t) = \frac{1}{2}(u_-(t) + u_+(t)),$$

where $s$ is the speed of the discontinuity and $u_\pm(t) = \lim_{x \to x_s(t)^\pm} u(x, t)$ and $s = \dot{x}_s(t)$.

**Solution**: Since the Burgers equation can be written as $u_t + (u^2/2)_x = 0$, the Rankine-Hugoniot condition for the position of the shock $x_s(t)$ gives

$$\frac{dx_s}{dt} = \frac{\frac{1}{2}u_+^2 - \frac{1}{2}u_-^2}{u_+ - u_-}, \tag{12}$$

where $u_+$ and $u_-$ are the values of $u$ to the right and to the left of the shock, respectively. The value to the left corresponds to characteristics emanating from $x_0 < 0$, for which $u = 1$, and the value to the left corresponds to characteristics emanating from $x_0 > 1$, for which $u = 0$ (a rough sketch of the characteristics might be useful here). Thus, $u_+ = 0$ and $u_- = 1$, and we have

$$\frac{dx_s}{dt} = \frac{\frac{1}{2}0 - \frac{1}{2}1}{0 - 1} = \frac{1}{2}. \tag{13}$$

Together with the initial condition $x_s(1) = 1$, we get $x_s(t) = 1 + (t - 1)/2$.

(c) Sketch the characteristics and the shock in the $(x, t)$ plane.

**Solution**: A sketch is shown below.

(d) Find the solution $u(x, t)$.

**Solution**: The solution satisfies the implicit equation $u = u_0(x_0) = u_0(x - ut)$. When $x_0 < 0$, $u_0 = 1$, and so we have $u = 1$ along the characteristics $x_0 = x - t$ for $x_0 < 0$, provided they haven't met the shock (blue lines in diagram). Similarly, $u_0 = 0$ for $x_0 > 0$, and so $u = 0$ along the characteristics $x_0 = x$ for $x_0 > 0$ (purple lines). Finally, if $0 < x_0 < 1$ we have $u_0 = 1 - x_0$, and so $u = 1 - (x - ut)$, which yields $u = (1 - x)/(1 - t)$ (green lines). Putting everything together, we obtain

3. **Wave Equation.** Consider the following initial-boundary value problem on the domain $D = \{(x,t) : t \in \mathbb{R}^+, x \in \mathbb{R}^+, x > t/\alpha\}$, where $\alpha > 1$:

$$u_{tt} = u_{xx}, \qquad x > t/\alpha,\ t > 0, \tag{15}$$

$$u(x,0) = \varphi(x), \qquad x > 0, \tag{16}$$

$$u_t(x,0) = \psi(x), \qquad x > 0, \tag{17}$$

$$u(x,\alpha x) = f(x), \qquad x > 0, \tag{18}$$

with $\varphi, \psi, f \in C^2(\mathbb{R}_0^+)$.

(a) Find the solution $u(x,t)$.

**Solution:** We seek a solution of the form

$$u(x,t) = F(x - t) + G(x + t) \quad (16)$$

(b) Find su□cient conditions on $\psi$, $\varphi$, and $f$ so that the solution is continuous in $D$.

Solution: We need to ensure continuity across $x = t$, where the two solutions meet. Letting $x \to t^+$ and using the fact that the functions involved are continuous we get

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Multiplying the PDE by $v$ and integrating over the domain, we have

$$\begin{aligned} 0 &= \int_{B(0,1)} v(x)\Delta v(x)\,dx \\ &= -\int_{B(0,1)} |\nabla v(x)|^2\,dx + \int_0^{2\pi} v(1,\theta)v_r(1,\theta)\,d\theta \\ &= -\int_{B(0,1)} |\nabla v(x)|^2\,dx, \end{aligned}$$

upon applying integration by parts and the boundary condition. Since the integrand is non-negative definite and the integral is zero, we must have

$$|\nabla v(x)|^2 = 0, \quad x \in B(0,1) \quad \Rightarrow \quad v(x) = const, \quad x \in B(0,1).$$

Since the average of $v(x)$ on the boundary is zero, $v(x)$ must be identically zero and uniqueness is proven.

(b) We seek a solution using the method of separation of variables in polar coordinates. Then, eq. (33) becomes

$$\begin{aligned} &u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad r \in (0,1), \quad \theta \in (0,2\pi), \\ &u_r(1,\theta) = g(\theta), \quad \theta \in [0,2\pi]. \end{aligned}$$

Seeking a solution in separated form $u(r,\theta) = f(r)g(\theta)$ implies

$$\begin{aligned} &g''(\theta) + \lambda g(\theta) = 0, \quad \theta \in (0,2\pi), \quad g(0) = g(2\pi), \quad g'(0) = g'(2\pi), \\ &f''(r) + \frac{1}{r}f'(r) - \frac{\lambda}{r^2}f(r) = 0, \quad r \in (0,1), \quad \lim_{r\to 0}|f(r)| < \infty. \end{aligned}$$

The angular boundary value problem has the trigonometric solutions

$$g_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta), \quad n = 0,1,2,\ldots.$$

with the corresponding eigenvalues $\lambda_n = n^2$.

The radial problem exhibits the bounded solutions

$$f_n(r) = r^n.$$

Introduce the series solution

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n[A_n\cos(n\theta) + B_n\sin(n\theta)].$$

The coefficients are determined by the boundary conditions

$$u_r(1,\theta) = \sum_{n=1}^{\infty} n[A_n\cos(n\theta) + B_n\sin(n\theta)] = g(\theta), \quad \theta \in [0,2\pi].$$

Multiplying by $\cos(m\theta)$ and integrating from 0 to $2\pi$, we obtain

$$A_m = \frac{1}{m\pi}\int_0^{2\pi} g(\theta)\cos(m\theta)\,d\theta, \quad m = 1,2,\ldots.$$

Multiplying by $\sin(m\theta)$ and integrating from 0 to $2\pi$, we obtain

$$B_m = \frac{1}{m\pi}\int_0^{2\pi} g(\theta)\sin(m\theta)\,d\theta, \quad m = 1,2,\ldots,$$

which determines a series representation of the solution. To determine $A_0$, we require zero average on the boundary so that $A_0 = 0$.

(c) Inserting the expressions for the coefficients into the series representation, we obtain

$$\begin{aligned} u(r,\theta) &= \sum_{n=1}^{\infty} \frac{r^n}{n}\frac{1}{\pi}\int_0^{2\pi} g(\varphi)\left(\cos(n\varphi)\cos(n\theta) + \sin(n\varphi)\sin(n\theta)\right) \mathrm{d}\varphi \\ &= \int_0^{2\pi} g(\varphi)\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{r^n}{n}\cos(n(\varphi-\theta))\,\mathrm{d}\varphi \\ &= \int_0^{2\pi} g(\varphi)N(r,\theta-\varphi)\,\mathrm{d}\varphi. \end{aligned}$$

where $a$ is a constant and the dot and prime indicate time and space derivatives, respectively. If $a = 0$, the spatial equation gives $X = A + Bx$, which upon evaluation of the boundary conditions leads to $X = 0$. Similarly, if $a > 0$ we get $X = Ae^{\sqrt{a}x} + Be^{-\sqrt{a}x}$, leading also to $X = 0$. Therefore, $a$ must be negative and we set $a = -\lambda^2$. We obtain

$$T(t) = T(0)\exp(-\lambda^2 t), \tag{42}$$
$$X(x) = A\sin(\lambda x) + B\cos(\lambda x). \tag{43}$$

Using the boundary conditions $X(0) = X(1) = 0$ we obtain $B = 0$ and $\lambda = n\pi$, so we get the modes

$$X_n(x) = \sin(\lambda_n x), \tag{44}$$

where $\lambda_n = n\pi$ and $n \in \mathbb{N}^+$. Thus, we find

$$\tilde{u}(x,t;s) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 t}\sin(\lambda_n x). \tag{45}$$

Using the initial conditions $\tilde{u}(x,t;s) = f(x)e^{-s}$ we get

$$f(x)e^{-s} = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 s}\sin(\lambda_n x), \tag{46}$$

which implies that $A_n = f_n e^{(\lambda_n^2-1)s}$, where $f_n$ is the $n$th sine Fourier coefficient of $f(x)$. Therefore,

$$\tilde{u}(x,t;s) = \sum_{n=1}^{\infty} f_n e^{(\lambda_n^2-1)s} e^{-\lambda_n^2 t}\sin(\lambda_n x). \tag{47}$$

and

$$u(x,t) = \int_0^t \tilde{u}(x,t;s)ds = \int_0^t \sum_{n=1}^{\infty} f_n e^{(\lambda_n^2-1)s} e^{-\lambda_n^2 t}\sin(\lambda_n x)ds \tag{48}$$
$$= \sum_{n=1}^{\infty} f_n e^{-\lambda_n^2 t}\sin(\lambda_n x)\int_0^t e^{(\lambda_n^2-1)s}ds \tag{49}$$
$$= \sum_{n=1}^{\infty} f_n e^{-\lambda_n^2 t}\sin(\lambda_n x)\left.\frac{e^{(\lambda_n^2-1)s}}{\lambda_n^2-1}\right|_0^t \tag{50}$$
$$= \sum_{n=1}^{\infty} f_n e^{-\lambda_n^2 t}\sin(\lambda_n x)\frac{e^{(\lambda_n^2-1)t}-1}{\lambda_n^2-1} \tag{51}$$
$$= \sum_{n=1}^{\infty} f_n \sin(\lambda_n x)\frac{e^{-t}-e^{-\lambda_n^2 t}}{\lambda_n^2-1}. \tag{52}$$

(b) Prove that the solution is unique.

**Solution:** Assume there are two solutions, $u_1$ and $u_2$. Then their difference $w = u_1 - u_2$ satisfies

$$w_t = w_{xx}, \qquad 0 < x < 1,\ t > 0, \tag{53}$$
$$w(x,0) = 0, \qquad 0 < x < 1, \tag{54}$$
$$w(0,t) = u(1,t) = 0 \qquad t > 0. \tag{55}$$

Let $T > 0$. By the maximum principle, the maximum of $w$ in the closure of $\overline{U_T} = [0, 1] \times [0, T)$ must be equal to the maximum of $w$ in its parabolic boundary, $\overline{U_T} - U_T$, which is zero. Therefore $w \leq 0$, or equivalently $u_1 \leq u_2$ in $\overline{U_T}$. Applying the same argument to $-w$ we conclude that $w = u_1 - u_2 \equiv 0$ in $\overline{U_T}$. Since $T$ was arbitrary, $u_1(x, t) = u_2(x, t)$ for all $t > 0$, $x$ $(0, 1)$, so the solution is unique.

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