

Program in Applied Mathematics
 PROBABILITY AND STATISTICS PRELIMINARY EXAMINATION
 January 11, 2019

Instructions Do four of the following six problems. Place an X on the lines next to the problem numbers that you are submitting for grading. Please do not write your name anywhere on this exam. You will be identified only by your student number. Write this number on each page submitted for grading. Show all relevant work.

1.
 2.
 3.
 4.
 5.
 6.
 Total

Student Number _____

1. Let $N(t); t \geq 0$, be a Poisson process with rate λ that is independent of the sequence of iid random variables $X_1; X_2; \dots$ with density $f(x)$.
- (a) Find the moment generating function of the compound Poisson process

$$Y(t) = \sum_{i=1}^{N(t)} X_i$$

- (b) Find the mean and variance of $Y(t)$.
2. Let $N(t); t \geq 0$, be a Poisson process with rate λ .
- (a) Find the covariance function for $N(t)$. That is, find $\text{Cov}(N(t_1); N(t_2))$ for arbitrary $0 \leq t_1 \leq t_2 < \infty$.
- (b) Let A_1 be the first arrival time of $N(t)$. Show that the conditional distribution of A_1 given $N(t_0) = 1$ is $U(0, t_0)$.
- (c) Show that $N(t)$ obeys the Markov property in that

$$E f(N(t)) | N(u) = s = E f(N(t)) \quad \text{for all } 0 \leq u \leq t$$

4. Suppose that $X_1; X_2; \dots; X_n$ is a random sample from the uniform distribution where $0 < X_i < 1$.

- (a) Find the maximum likelihood estimator (MLE) for θ based on $X_1; X_2; \dots; X_n$.
- (b) Is this estimator for θ unbiased? Justify your answer.
- (c) Now suppose that we do not get to observe $X_1; X_2; \dots; X_n$ but instead we observe $Y_1; Y_2; \dots; Y_n$ where

$$Y_i = \begin{cases} 1 & ; \text{ if } 0 < X_i < 1 \\ 0 & ; \text{ otherwise} \end{cases}$$

Find the MLE for θ based on $Y_1; Y_2; \dots; Y_n$.

- (d) Compute the relative efficiency of these two estimators of θ .

5. An ecologist studying a particular population of grasshoppers measures the height (in cm) of grasshoppers' initial jump under a replicable set of conditions in the field. A combination of empirical study and theoretical considerations has caused the ecologist to assume that each of these $H_1; \dots; H_n$ are independent random samples from a Normal distribution with mean μ and standard deviation 1 cm. You should assume this too.

- (a) Show under what conditions for $a_1; \dots; a_n$; where $a_i \in \mathbb{R}$, that $W = \sum_{i=1}^n a_i H_i$ will be an unbiased estimator of μ .
- (b) Find the unbiased estimator of this form that has minimum variance. What is the variance of this estimator?
- (c) Is this estimator the UMVUE? Show your work for full credit.
- (d) Now consider that the ecologist wants to estimate the probability that a grasshopper from the same population (under the same field conditions) will jump higher than 10 cm on its initial jump. Find the MLE of this probability. You can express your answer in terms of the c.d.f. of the standard Normal distribution, $\Phi(\cdot)$.
- (e) For her research, the ecologist wants to know if the jumping behavior of the same population of grasshoppers changes during a total solar eclipse. She measured 6 new grasshoppers' jumps during the latest eclipse. Recognizing that the field conditions had changed (the eclipse being a total solar eclipse), she found the MLE of μ to be 10.9 cm.