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Big Ideas in the Understanding of Fractions: A Learning Progression

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For some time now, *learning progressions* (LPs), which are descriptions of the developmental path that students are likely to take when learning the concepts in a given domain (Clements & Sarama, 2004), have been viewed as a promising means of coordinating three elements that are critical to student learning: curriculum, instruction and assessment (Clements & Sarama, 2004; Lobato & Walters, 2017). By emphasizing the multiple-choice tests for this purpose[1]. The *i-Ready Diagnostic*, for example, provides a list of

- each student s assessment scores that are meant to help

. While all instances of formative assessment, including these large-scale assessments, present opportunities for both providing feedback to students and adjusting instruction, the more immediate the adjustment, the stronger the impact on student learning is likely to be. There may, therefore, be a disconnect between the kinds of local, teacher- and researcher-developed assessments that are most frequently used in the development and validation of LPs (e.g.,

Norton, 2018; Wright, 2014; Yulia et al., 2019), and the large-scale interim and summative assessments that are commonly used by teachers in K-12 public school settings in the United States. The former tend to allow for more immediate and iterative feedback during the learning process, while the latter are typically only administered a few times a year after learning has occurred.

In this paper we address this disconnect by using the results from a widely used

i-Ready Diagnostic, to support an LP for fractions. We present compelling empirical evidence to support the validity of this LP by taking advantage of *i-Ready*

designed for students in different grade levels. By using the vertical scale to investigate the relative difficulties of items that are more and less mathematically sophisticated, according to the LP, we provide an example where an LP can provide a productive structure for making sense of results of an externally mandated assessment. We note up front that although the research

[1] Three of the commercial assessments commonly purchased by U.S. school districts for this purpose are the MAP Growth Assessments (NWEAS x8(N)] TJETQq0.00000912 0 612 792 reW* nBT/F43W* nBT/F1 1 reW* nBT/F1 12 Tf1 0 0 1 116.3

presented in this study did occur within the context of a collaboration with Curriculum Associates that was intended to contribute to their ongoing efforts to improve upon their assessment and curricular products, our research team was given full autonomy in the development and validation efforts related to an LP for fractions.

In what follows, we begin by situating our study within the curriculum and assessment context of the *i-Ready Learning* and *i-Ready Diagnostic* products that Curriculum Associates provides to school districts in the United States. We motivate our specific focus on an LP for the understanding of fractions, and then provide a summary of the results from our review of pre-existing fraction LPs. We then introduce an LP that takes a slightly larger scope and grain-size than do the LPs found in the pre-existing mathematics education literature. Our intention in using this grain-size is to provide teachers with a handful of large categories that they can hold in their minds more effectively than a series of hyper-specific standards. Furthermore, it is easy to get 90QEW*0 1 248.6 570.60 g0 bogged down in specifics and to focus on individual skills rather than on the concepts that underly those skills. Our intension in this work is to encourage teachers to focus on the big ideas

facts and skills to be memorized and reproduced. Since our 57>7<0003005200490085 Tm0 g0 G(i)7(on i179r2E

LP level. We conclude with a discussion about how the LP can be used to support formative inferences about student learning in classroom contexts that feature a combination of locally developed and externally mandated assessments.

The Curriculum and Assessment Context

The *i-Ready Diagnostic* assessment comprises two grade-specific standardized tests in reading and mathematics that are intended to be administered during the fall, the winter and the spring of each academic school year. Students take each test on a digital interface, and tests are designed to be adaptive such that each new multiple-choice item to which a student is exposed depends upon whether they answered prior items correctly. The mathematics test for students in grades K-12 consists of up to 66 items, and the content of these items is organized into four strands: Algebra, Geometry, Measurement, and Number and Operations. The test was developed to serve the following four purposes (Curriculum Associates, 2018, p. 8):

1. Establish a metric that will allow for an accurate assessment of student knowledge that can be monitored over a period of time to gauge student improvement.
2. Accurately assess student knowledge for different content strands within each subject.
3. Provide information on what skills students are likely to have mastered and likely need to work on next.
4. Link the assessment results to instructional advice and student placement decisions about *i-Ready Instruction* curricula and print products.

i-Ready Learning, a library of online instructional modules, and Ready Learning, a series of grade-specific printed books containing instructional lessons.

Curricular priorities: The importance of fractions

The reasoning skills that students typically develop with regard to fraction operations and equality in grades three through six help to build a sense of mathematical structure that facilitates the learning of more formal algebraic concepts later on (Common Core State Standards Initiative, 2010; Empson et al., 2011). In particular, students need to understand conceptually what fractions are and how they interact with one another (Byrnes & Wasik, 1991), which largely involves seeing fractions as real number values that can be placed at a unique point on a number line (Hansen et al., 2015; Siegler et al., 2011) and developing proportional reasoning and visualization skills (Hansen et al., 2015). There is also empirical evidence that students with a solid understanding of fractions are more likely to be successful in future mathematics coursework in middle school and beyond (Bailey et al., 2012; Booth & Newton, 2012; Siegler et al., 2012; Torbeyns et al., 2015).

Given the foundational nature of fractions understanding, we developed a learning progression for fractions that is meant to help students, teachers, and parents track student growth in this domain. During the development process, we reviewed existing LPs and the

as well as the Common Core State Standards for Mathematics (CCSS-M) and the curricular focus and ordering of the fractions-related content in the *i-Ready* curriculum.

A review of pre-existing learning trajectories for fractions

conceptualize fractions. LP, students move through different levels of each conceptualization starting with (1) unit forming, followed by (2) unit coordinating, (3) equivalence, and (4) comparison. As we will describe in the following section, we take the four fraction conceptualizations to be part of our LP, and we argue that the four levels Wright presents actually map onto those conceptualizations.

Wilkins and Norton (2018) published another fractions LP in which they proposed a hierarchy among the fraction schemes that Steffe and Olive (2010) put forward in their foundational book t of fraction understandings. Wilkins and Norton developed a test with items designed to measure each of the schemes of interest (four items per

The final LP that we identified was produced by Yulia and colleagues (2019). They reviewed the extant literature to develop a hypothetical LP and associated tasks, which they checked with teachers participating in their study, then revised as needed. Validation of this LP consisted of classroom observations and task-focused cognitive interviews with 25 students in Indonesia. Their LP order is (1) fractions as a part-whole relationship, (2) determining fractional equivalence, (3) comparing fractional values, and (4) operating with fractions.

The four LPs described above each provide valuable conjectures regarding the paths that students tend to take when learning about fractions. However, each LP features either a level or developmental conceptualization of fractions that is unique to that study. For example, the concepts of ratios and rates 014) work, and even then, they are treated as their own distinct construct through which students move rather than as an ordered level within a larger fractions construct. Furthermore, two of the LPs used qualitative methods with fairly small samples in local contexts
2019). It is also an open question whether the order implied by these pre-existing LPs has been sufficiently validated. For example, the LP by Wright (2014) used a small sample of only six students in one classroom and was unclear in describing how the patterns in test scores were identified. The LP by Wilkins & Norton (2018) had a larger and more geographically-diverse sample, but the assessment they used only contained four items for each of the four schemes that they investigated.

A New Learning Progression

form a coherent learning progression that they can be viewed as ordered levels of a single construct and that the level of sophistication with which a student can understand fractions and use this understanding to solve mathematical problems increases from one conceptualization to the next. The levels of the expected progression through this construct (from lowest to highest) are described in the following subsections and are also summarized in Table A-1 in the Appendix.

Level 1: Fractions as parts of a whole

The conceptualization that has historically been used to introduce students to fractions is the *part-whole conception* in which a whole is partitioned into equal parts and some of those parts are mentally disembedded (Moss, 2005; Steffe & Olive, 2010). This can take the form of one object being split into multiple parts, as in an area model in which a pizza, for instance, is cut into equal slices. Alternatively, it may consist of a set of objects, some fraction of which are specified (e.g., if there are three people sitting at a table set for four, then three-fourths of the seats are occupied). This conceptualization is representative of the first level in three of the four existing LPs that we identified

Yulia et al., 2019). Figure 1 is an example of a set model. In this item, there are five smiley

Figure 1. Example of a part-whole item

faces, and the students are asked to identify which fraction of the whole set is shaded. In this case, sense by counting wholes) (Moss, 2005). Students working within a part-whole framework may maintain their natural number context and apply the counting strategies that have served them well in other situations (Post et al., 1993).

The part-whole conception of fractions is insufficient on its own for students to develop a complete understanding of fractions and the ways that they may be used (Hackenberg & Lee, 2015; Post et al., 1993; Steffe & Olive, 2010). In particular, students who can identify both the total and the specified number of pieces in a fraction may not fully comprehend that all of the pieces must be equal or that the all of the original whole must be used when creating fractional pieces

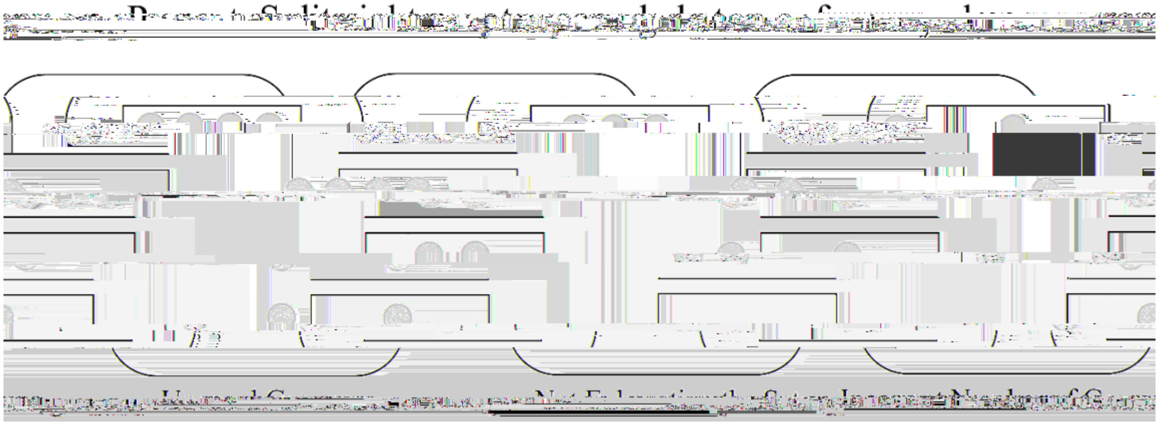

Figure 2. Examples of inaccurate attempts at fair sharing

claiming that $\frac{10}{15} > \frac{2}{3}$ because $10 > 2$ and $15 > 3$ (Hart et al., 1981). These applications of natural-number logic may lead to difficulties adding and subtracting fractions, even those with the same denominator, if students overgeneralize traditional operation procedures (e.g., adding across both the numerators and the denominators) (Newton, 2008; Post et al., 1993; Wu, 2001). In order to move past these obstacles, students must be able to view fractions as representing an equipartitioning process in which a whole is divided into equal parts.

Level 2: Fractions as quotients

(Clarke, 2011) that a fraction is a quotient such that – represents the division of by . The key to this stage of understanding is the ability to engage in

al., 2012). In fair sharing, the number of individual units to be shared ( ) is divided by the number of shares that are needed ( ) such that the size of each share is – (- ths) of one unit (Empson et al., 2006). The foundational example of this process is taking one unit and dividing it into

objects were a multiple of the number of shares needed, as in the coin-sharing problem described

values and need to create their own (Moss & Case, 1999). If a student, for example, were asked to locate $\frac{3}{8}$ on the number line shown in Figure 4, they would need to split the areas between the one-fourth markings in half to create eights.

A measurement conception of fractions is most important when students need to compare relative magnitudes of fractions with unlike numerators and denominators, usually in a context where they need to determine whether one fractional value is greater than, less than, or equal to another (Steffe & Olive, 2010). If a student does not see a fraction as representing a specific, orderable value, then they will 00912 tific, Students need an accurate mental number line (Griffin, 2004; Hamdan & Gunderson, 2017) and a sense of how quantities combine to form new quantities (Jordan et al., 2007) in order to develop fraction addition skills (Keijzer & Terwel, 2003). Having a sense of magnitude and additive properties allows students to reject implausible answers and the procedures that they used to obtain them (Booth & Siegler, 2008; Hiebert & Lefevre, 1986; Siegler et al., 2011). Understanding magnitude and number lines is also essential for comprehending the meaning of improper fractions. If a student were to only make use of a part-whole conception, it would make no sense to have more pieces than there are meant to be i they

pineapple juice.

Misunderstanding the *direction of effects* of fraction operations indicates that students may have simply memorized algorithms and do not yet fully understand what fraction multiplication and division mean conceptually. This is likely why students who have not mastered the operator conceptualization of fractions tend to have difficulty selecting procedures when solving proportion problems that are not simply presented as a symbolic missing value, such as $\frac{2}{7} = \frac{6}{}$. Figure 6 presents one such problem. In this item, three friends are sharing the left-over quarter of the cake such that they each get $\frac{1}{4}\ \ \frac{1}{3} = \frac{1}{12}$ of the original whole. This item requires students to recognize that they need to divide the left-over quarter of the cake into three equal pieces, which is the same as multiplying by $\frac{1}{3}$. Similar items may change the context and the proportions involved. A more difficult iteqETQq(r i)7(t)hange tude6f1 0 0 more diffic d and>792 reW*7(ore)

Misunderstanding the 4theext and

While all the existing LPs have levels that fit well within our defined levels, they are either missing one or more of the conceptualizations Kieren identified or are too focused on individual skills for our purposes. The LP proposed by and (2014) does have conceptually-based levels that cover the big ideas of a unit, fractions as numbers, and the additive and multiplicative structure of fractions, but it does not include levels that address

LP s

(2014) LP

Table 1. Mapping our LP onto existing LPs

LP Level	CCSS-M (2010)	A-A & C-H (2014)	Wright (2014)	W & N (2018)	Yulia et al. (2019)
Operator	6.NS.1 5.NF.4-7	(4) Multiplicative structure			(4) Operations of fractions
Ratio	6.RP.1-3				
Measurement	5.NF.1-2 4.NF.1-6 3.NF.2-3	(3) Additive structure (2) Fraction as number	(4) Comparison (3) Equivalence	(4) Reproduce a wp nE99.35 378.65	

Situating our learning progression

The learning progression that we have defined uses a combination of approaches to LP design, as described by Lobato and Walters (2017). Our approach is cognitive in the sense that we identify a series of increasingly sophisticated conceptions and in the conjecture that students develop these more sophisticated conceptions about fractions over time. This type of LP typically uses cross-sectional data across several grade levels to help validate this conjecture, which is also the approach we take in our use of the *i-Ready Diagnostic* assessment data. Our approach is also based upon disciplinary logic and curricular coherence in that it *informed by research* versus being the *product of research* (emphasis in original, Lobato & Walters, 2017, p. 87). The

We conceptual understanding of fractions as a location on a latent continuum that is assumed to remain relatively stable as a student is completing assessment items on any given occasion, and argue that a location on this continuum can be inferred (with some amount of measurement error) through an analysis of their pattern of item responses.

Validation Approach

In order to empirically test our hypothesized learning progression, we examined the relationship between our ordering of the levels of fractional knowledge described in the previous sections and the difficulty estimates of the *i*-

would have a 50% chance of giving a correct answer to an item with a difficulty rating of 350. More specifically, Curriculum Associates uses a vertical scale in which scores from assessments administered to non-equivalent groups of examinees, such as students in different grades, are placed onto a single scale so that, for example, a third grader and a fourth grader with the same numeric score on the *i-Ready Diagnostic* are interpreted as having demonstrated the same level of absolute proficiency despite having taken different diagnostic test items (for an overview of vertical scaling, see Tong & Kolen, 2010). For the *i-Ready* tests, this was done by administering common items across grade levels, and then using the information about student

Comparisons based upon t

the estimated difficulties of the *i-Ready Diagnostic* items that assess the content covered in those lessons.

Content experts at Curriculum Associates coded the items in the *i-Ready Diagnostic* assessment system according to the specific content knowledge and skills that students are presumed to require to accurately complete the problem. Curriculum Associates refers to these codes as anchor claims. We identified 406 fractions items associated with 107 anchor claims. The first and third authors independently coded each anchor claim based on which of the five fraction conceptualizations from our LP would be most important to understand in order to

Table 2. Illustrative anchor claims and associated Ready Lessons[3]

LP

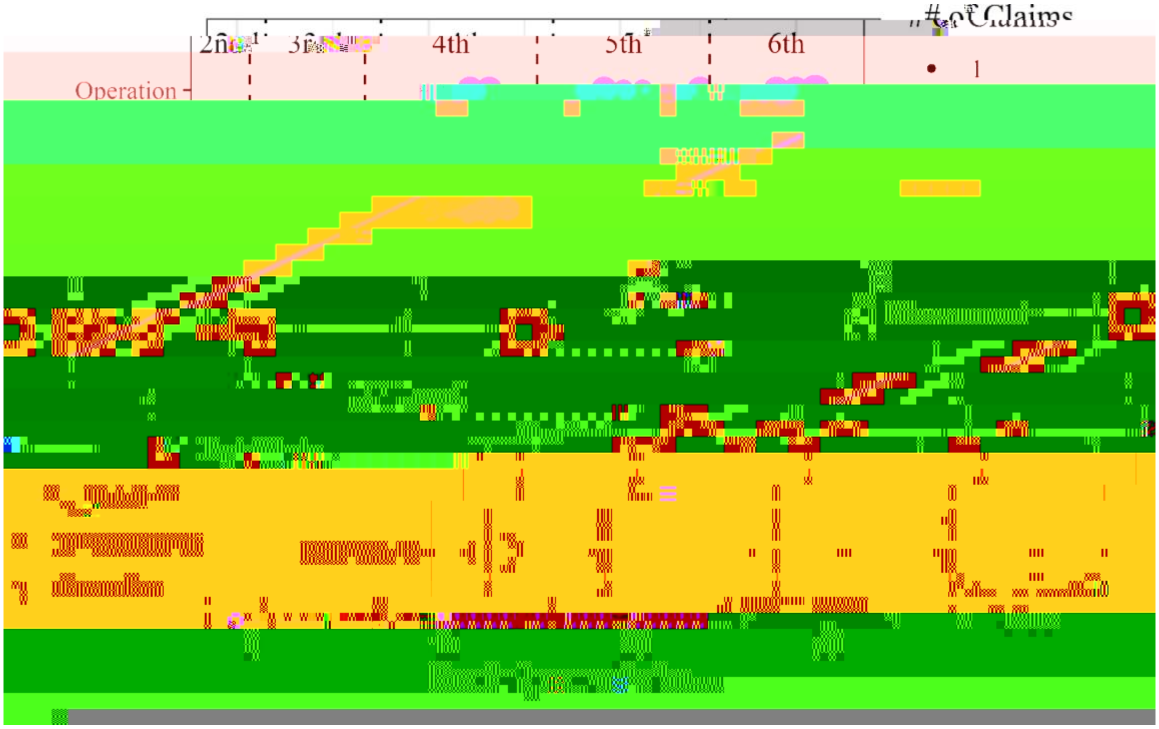

Figure 7. Anchor claim LP levels by the order in which they appear in the Ready lessons

by the Quotient, Measurement, Operator, and Ratio conceptualizations. The conceptual focus also shifts between the grades such that second grade is Part-Whole only and third grade introduces Quotient and Measurement concepts. Fourth grade is mostly Measurement and introduces Operator ideas. By fifth grade, students are mostly learning Operator concepts, and Ratios are introduced in sixth grade, with Operator ideas still strongly featured. There is an upward trend in the regression line shown in Figure 7, which was generated by coding the LP levels from 1-5 (part-whole = 1 and operator = 5). Because of the ordinal nature of our analysis, w to examine the association between the LP levels and the lesson ordering using this numerical coding scheme. The value of was 0.68. This indicates a fairly strong relationship between the levels of fraction understanding that we have defined and the curricular ordering of the Ready curriculum. The correspondence would likely be even stronger if Ratios were introduced sooner, as our evidence points to Ratio items being easier than Operation items in general despite being introduced later.

We can see in Table 3 and Figure 8 that the difficulties of the *i-Ready* assessment items associated with anchor claims tend to increase when the conceptualizations are placed in the order in which they appear in our LP value of 0.55. The difficulties of the items associated with Ratios tend to be much more dispersed relative to those of the adjacent Measurement and Operator conceptualizatio51 203.5nBT/.80 0 1 248.85 597.9BT/F1 12 Tf1 0 0 1 160.58 542.4

range of difficulties because of the Part-Whole aspects of this conceptualization, which are more similar to the lower-level conceptualizations. It is when the items move into the part-part rate ideas that the difficulties increase. This finding also aligns with previous research that has shown that students tend to begin building informal understandings of ratios and proportion fairly early, but it often takes quite a while for the formal ideas to develop (Bruner et al., 1966).

We ran an ANOVA to compare the means of the items in the LP level groups, and found that the differences in means were statistically significant [$F(4, 401) = 52.27$, $p < .001$]. We then

Using a Learning Progression for Formative Assessment

Our analyses support the existence of a five-level, course-grained learning progression for fractions that, except for the Ratios conceptualization, aligns with the curricular ordering found in the CCSS-M (CCSSI, 2010) and in the *Ready* and the *i-Ready* curricular programs. This suggests that it is possible to use evidence from the large-scale diagnostic assessments that teachers are often required to administer to develop an LP that is consistent with existing LPs in the research literature that were developed using more locally developed assessments.

LPs have the potential to serve as powerful assessment tools, as they may be used formatively by teachers during their informal interactions with students and in more formal classroom assessments that they may create (Clements et al., 2011; Clements & Sarama, 2008; Edgington, 2014; Furtak et al., 2014). Our intention in using a large grain-size for the LP was to allow teachers to more easily internalize a handful of levels that could serve as guideposts

There are, however, some conditions that must be met for LPs to be used effectively. First, the LP must accurately represent the typical ordering that students experience when

This means that there may be items that were classified as representing aspects of a given level that also included other constructs. This could have caused construct-irrelevant variance and skewed the difficulties of some items.

Another potential concern is whether our selected grain-size is useful to teachers in the way we expect it to be. Our team will be conducting interviews with teachers who use the *i-Ready Diagnostic*, and we intend to ask them whether these four levels make sense to them and would be helpful to guide their thinking and classroom practice.

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Appendix

Table A-1. Fraction conceptualizations learning progression

Interpretation	Student Characteristics	Item Responses
Operator	**Understands that:** x Multiplying a value by a fraction – results in a value that is - ths of the original value x Understands the difference between multiplying and dividing fractions	**Is able to:** x Use multiplication to find a portion of a value x Determine that multiplying a value by a fraction with magnitude less than 1 will result in a value with smaller magnitude and multiplying by an improper fraction will result in a value with larger magnitude, and vice versa for division, without performing the calculations x Divide a value by a fraction
Ratio	**Understands that:** x Ratios may be expressed in various forms (–, : , verbal description, or diagram) x Ratios may represent either part-whole or part-part relationships x Ratios may represent rates x Equivalent ratios may be created by multiplying both parts by the same value **May not yet understand that:** x Multiplying a rate by a value can provide information about the overall situation (e.g., if a driver goes $\frac{65\text{ miles}}{\text{hour}}$ for 3 hours, they have gone $\frac{65\text{ miles}}{\text{hour}} \times 3\text{ hours} = 195$ miles) x The direction of effects for fraction operations are not the same as they are for whole numbers	**Is able to:** x Identify part-whole and part-part relationships x Move between the various representational forms for ratios and rates **Common Errors:** x Selecting the wrong operation when solving problems involving proportional reasoning x Indicating that multiplication always results in a larger value and that division always results in a smaller value

Measurement

Understands that:

- Fractions represent unique numerical values
- Two fractions are equivalent if they represent the same numerical value
- Fractional values can be converted to decimals or percentages while maintaining their numerical value
- Improper fractions may be rewritten as mixed numbers and vice versa
- Fractions with different denominators may be compared or added if they are put into the same units

May not yet understand that:

- Fractions may be written as ratios and may represent part-part relationships or rates

Is able to:

- Create and identify equivalent fractions, including converting between improper fractions and mixed numbers
- Order fractions and mixed numbers with different numerators and different denominators
- Add and subtract fractions and mixed numbers with different denominators

Common Errors:

- Treating all ratios as part-whole
- Treating rates as two independent values with different units

Quotient

Understands that:

- not appear the same
- The fraction – represents the division of by
- Unit fractions can be iterated to reproduce the original whole or part of the whole
- Dividing the same whole into more parts (larger denominator) results in smaller unit pieces

May not yet understand that:

- A fraction has its own specific value that can be uniquely placed on a number line.

- Incorrectly comparing two fractions with different numerators and different denominators
- Not recognizing improper fractions as valid

Part-Whole

Understands that:
- A fraction represents a specified number of parts out of the total number of parts

May not yet understand that:
- A whole must be partitioned equally
- All parts of the whole must be used when partitioning

Is able to:
- Identify the number of specified and total parts in an area model or in a described situation.
- Compare fractions with the same denominator and different numerators

Common Errors:
- Making unequal parts or fail to exhaETQq148.3 31 12